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UNIFORMLY DISTRIBUTED SEQUENCES OF p–ADIC INTEGERS, II

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Abstract. The paper describes ergodic (with respect to the Haar measure) functions in the class of all functions that are defined on (and take values in) the ring $\mathbb{Z}_p$ of p-adic integers, and satisfy (at least, locally) the Lipschitz condition with coefficient 1. Equiprobable (in particular, measure-preserving) functions of this class are described also. In some cases (and especially for $p = 2$) the descriptions are given by explicit formulae. Some of the results may be viewed as descriptions of ergodic isometric dynamical systems on the p-adic unit disk. The study is motivated by the problem of pseudorandom number generation for computer simulation and cryptography. From this view the paper describes nonlinear congruential pseudorandom generators modulo m that produce stricly periodic uniformly distributed sequences modulo m of a maximum possible period length (i.e., exactly m). Both state update functions and output functions of these generators could be, e.g., meromorphic on $\mathbb{Z}_p$ functions (in particular, polynomials with rational, but not necessarily integer coefficients), or compositions of arithmetical operations (like addition, multiplication, exponentiation, raising to integer powers, including negative ones) with standard computer operations, such as bitwise logical operations (e.g.,XOR, OR, AND, NEG, etc.). The linear complexity of the produced sequences is also studied.

1. Introduction.

A number of applications in computer simulation, numerical analysis (especially Quasi Monte Carlo) and cryptography demand regular methods to generate successively a uniformly distributed sequence. The corresponding literature is so vast that we could not even mention here the most important monographs in the area. We refer only [2], where a reader could found a rather substantial survey of relevant methods as well as a comprehensive bibliography. The major part of these methods are certain recursive procedures, which may be viewed also as automata. The latter are commonly referred as pseudorandom (or quasirandom) generators.

The typical one is the so-called linear congruential generator, which has been developed more than half a century ago. It produces a sequence $\{x_n : n = 0, 1, 2, \dots\}$ over a set $\{0, 1, \dots, m-1\}$ (the latter is commonly treated as the residue class ring $\mathbb{Z}/m$ of the ring $\mathbb{Z}$ of rational integers modulo natural $m > 1$), which is a first order recurrence sequence, defined by $x_{n+1} \equiv a + bx_n \pmod m$ with integer rationals

[1]The current version (Dec., 2004) of the paper slightly differs from the published one: Some typos were fixed, some improvements in English were made.

Key words and phrases. Uniformly distributed sequence; p-adic integer; non-Archimedean dynamical system; ergodic function; equiprobable function; measure-preserving function; transitive polynomial; bijective polynomial; permutation polynomial; pseudorandom number generator; nonlinear congruential generator; linear complexity.

a, b. The sequence is uniformly distributed iff it is purely periodic and its shortest period is of length m. The latter condition implies that *each* element of $\mathbb{Z}/m$ occurs at the period *exactly once* and vice versa. The necessary and suffcient conditions a and b must (for the given m) satisfy to provide the maximum period length (i.e., m) of the produced sequence, are well known –see [2, section 3.2.1.2, theorem A].

The undoubtful advantage of linear congruential generators is the simplicity (especially for $m = 2^k$) of their program implementations. One of the key reasons of their disadvantages (e.g., lack of statistical quality of the produced sequences, for certain applications) is their linearity. For instance, as the state update function $f(x) = a + bx$ of the generator is a polynomial of degree 1, the produced sequence has linear complexity 2 over the ring $\mathbb{Z}/m$, i.e., it is a linear recurrence sequence of order 2 over $\mathbb{Z}/m$ (defined by $x_{n+2} \equiv (1+b)x_{n+1} - bx_n \pmod m$). Hence, *for each* m the points $(\frac{x_{n+2}}{m}, \frac{x_{n+1}}{m}, \frac{x_n}{m})$ fall into the parallel planes $c + X - (1 + b)Y + bZ$ ($c \in \mathbb{Z}$), which intersect the unit cube of Euclidean space. The well known result due to George Marsaglia [7] states that similar effect also holds in higher dimensions > 3: All the points fall into the relatively small number of parallel hyperplanes (rather than fill this cube more or less uniformly), and the reason is again that $\deg f = 1$.

During the past decades these considerations stimulated the developement of various alternatives to linear congruential generators. The significant part of these are *nonlinear* congruential generators with the state update function f being either a polynomial over $\mathbb{Z}$ of degree > 1, (in particular, quadratic [2], or of higher degree [15]), or some non-polynomial transformations, e.g., exponential generators (with $f(x) = a^{g(x)}$), or the so-called inversive generators, which involve raising to negative powers (for the survey of different generators we again refer to [2]). Very often some authors seem to be more concerned with the linear complexity of the produced sequence than with its uniform distribution, admitting non-maximal period length, i.e., they admit state update functions f, for which the sequence never attains period length m, and hence, in a strict sence, is not uniformly distributed in $\mathbb{Z}/m$. In such cases the authors have not only to estimate possible period lengths, but also they have to choose the initial state (the *seed*) x_0 of the generator according to certain (sometimes, sophisticated) procedures that guarantee sufficiently long period, rather than to choose the seed at random.

Increasing the degree of a polynomial (as well as the use in the composition other arithmetical operations like exponentiation or taking an inverse) leads to increased complexity of program implementation. Usually there is a trade-off between the statistical quality and the complexity of the program implementation: The better the quality the slower the performance; fast generators sometimes demonstrate lack of quality.

So it is still important to find new classes of functions $f\colon \mathbb{Z}/m \to \mathbb{Z}/m$ such that the corresponding generators

(1) guarantee a maximum possible period length (i.e., exactly m) of the recurrence sequence of states defined by the relation $x_{n+1} \equiv f(x_n) \pmod m$, hence providing uniform distribution of this sequence over $\mathbb{Z}/m$ (we call these transformations f *transitive modulo* m);
(2) guarantee suffuciently large linear complexity over $\mathbb{Z}/m$ of the produced

sequence, i.e., absence of 'short' (in some definite sence) linear dependencies of the form $\sum_{i=0}^{r-1} c_i x_{n+i} \equiv 0 \pmod m$ $(n = 0, 1, 2, \dots)$ among the members of the sequence;

(3) are 'easy-to-implement', namely, are enough 'flexible', that is, have certain (crucial to the performance) parameters, varying which it is possible to gain speed without losses in quality.

The paper presents wide classes of transformations f that (to some extend) satisfy these conditions.

At the first turn we obtain conditions that guarantee transitivity modulo m for functions that are compositions of arithmetical operations (addition and multiplication of integers), as well as of standard computer ones, like bitwise logical operations, shifts, masking, etc. These compositions also might involve exponentiation and taking a multiplicative inverse, hence, raising to negative powers (see 4.9, 4.11, 2.5) and/or $\mathrm{OR}, \mathrm{XOR}, \mathrm{AND}$, etc., see 2.5, 2.8.

In particular, we describe wide classes of transitive modulo m functions that could be expressed as integer-valued polynomials with rational (and not necessarily integer) coefficients (see 4.7), as well as by analytic functions (4.11, 4.9, 2.5) or meromorphic (in particular, raional) functions (4.9, 4.11, 4.12). These conditions are easy-to-verify, and with the use of them the various explicit formulae for transitive modulo m transformations could be (and are) obtained – see e.g. 2.3, 2.4, also 2.5–2.8 (as well 4.11, 4.12) together with 2.1, and other examples here and there in the paper.

To illustrate, we start with some of these examples: Theorem 2.7 together with lemma 4.11 imply that each transformation f of the form

$$f(x) = 1 + x + 2(g(x+1) - g(x))$$

is transitive modulo $m = 2^k$ *for all* $k = 1, 2, \dots$ and for *arbitrary* composition g of

(1) arithmetical operations — an addition $(y, z) \mapsto y + z$, a multiplication $(y, z) \mapsto yz$, an exponentiation $(y, z) \mapsto (1 + 2y)^z$ (in particular, taking an inverse $y \mapsto (1 + 2y)^{-1}$),

(2) bitwise logical operations — such as conjunction $(y, z) \mapsto y \,\mathrm{AND}\, z$, disjunction $(y, z) \mapsto y \,\mathrm{OR}\, z$, exclusive 'or' $(y, z) \mapsto y \,\mathrm{XOR}\, z$, negation $z \mapsto \mathrm{NEG}\, z$, etc.,

(3) machine operations (which could be derived from the bitwise logical ones) — an s-step shift towards most significant bits $z \mapsto 2^s z$, masking $z \mapsto z \,\mathrm{AND}\, M$, M being a mask, 'reduction modulo 2^s', i.e., a truncation of the most significant bits $z \mapsto z \bmod 2^s = z \,\mathrm{AND}(2^s - 1)$, and some others.

We assume here that all the operands are non-negative integer rationals represented by their base-2 expansions; so, for instance, $2 = 1 \,\mathrm{XOR}\, 3 = 2 \,\mathrm{AND}\, 7 \equiv \mathrm{NEG}\, 13 \pmod 8$, $3^{-1} \equiv 11 \equiv -5 \pmod{16}$, $3^{-\frac{1}{3}} \equiv 3^{11} \equiv 3^{-5} \equiv 11 \pmod{16}$, etc. Up to this agreement the functions g and f are well defined on $\mathbb{Z}/m$, the complexity of their program implementation depends only on the number of 'fast' and 'slow' operations in the composition g and hence one may vary it in a wide range to achieve the desired performance.

We emphasize, in the example just mentioned transitivity modulo $m = 2^k$ of the function f *does not depend* neither on k nor on actual form of the composition g

— both for $g(x) = x \,\mathrm{XOR}(2x+1)$ and for

$$g(x) = \left(1 + 2\frac{x \,\mathrm{AND}\, x^2 + x^3 \,\mathrm{OR}\, x^4}{3 + 4(5+6x^5)^{x^6 \,\mathrm{XOR}\, x^7}}\right)^{7+\frac{8x^8}{9+10x^9}}$$

the sequence $\{x_n\}$ defined by the recurrence relation $x_{n+1} \equiv 1 + x_n + 2(g(x_n + 1) - g(x_n)) \pmod{2^k}$ is uniformly distributed in $\mathbb{Z}/2^k$ for each $k = 1, 2, 3 \dots$. Actually, this sequence is strictly periodic with period length 2^k, and *each* element of $\{0, 1, \dots, 2^k - 1\}$ occurs at the period *exactly once*.

Similar assertions also hold for arbitrary composite m: e.g., 4.11 and 4.12 imply that the transformation

$$f(x) = 1 + x + \pi(m)^2 u(x)(1 + \pi(m)v(x))^{w(x)}$$

with $\pi(m)$ being a product of all prime factors of m, is transitive modulo m for arbitrary polynomials $u(x), v(x), w(x) \in \mathbb{Z}[x]$ over $\mathbb{Z}$. A variety of results of these kind may be obtained in much more general situation for integer-valued polynomials with rational (not necessarily integer) coefficients by applying the techniques of section 4.

Note that this example also demonstrates how by minor changes of the recurrence relation one may achieve the transitivity of both inversive generator (for which $f(x) = a + bx^{-1}$ or $f(x) = (a + bx)^{-1}$) and exponential generator (with $f(x) = a^x$): for $w(x) = const = -1$ the introduced generator is of inversive type, for $v(x) = const \neq 0$ it is of exponential type.

As for linear dependencies $\sum_{i=0}^{r-1} c_i x_{n+i} \equiv 0 \pmod{m}$ $(n = 0, 1, 2, \dots)$ of fixed length r in produced sequences $\{x_n \equiv f(x_{n-1}) \pmod{m} : n = 1, 2, \dots\}$, one may say that from this view among all congruential generators linear ones are rather exceptions than the law. For instance, if $f\colon \mathbb{Z} \to \mathbb{Z}$ is represented by a transitive modulo some prime power $m = p^k$ $(k \geq 3)$ polynomial of degree ≥ 2 with integer rational coefficients, no such dependencies with r *and* c_i *not depending on* k do exist. Moreover, in this case the minimal order of linear recurrence sequence over $\mathbb{Z}/p^k$, which represents the produced sequence, tends to infinity together with k (in fact, much more general result holds — see 5.1–5.4 for exact statements).

The paper also studies equiprobable modulo m functions, i.e., mappings F of the sth Cartesian power $(Z/m)^{(s)}$ onto the tth Cartesian power $(Z/m)^{(t)}$ of the ring $\mathbb{Z}/m$, $(s \geq t)$, such that all preimages of all elements are of the same cardinality. In particular, for $s = t$ equiprobable modulo m functions are bijections of the corresponding rings and throughout the paper are referred as bijective modulo m functions. A very particular case of the equiprobable modulo m functions studied here are so-called permutation polynomials modulo m, the latter being polynomials over $\mathbb{Z}$ which induce bijections of the ring $\mathbb{Z}/m$ onto itself. The results of the paper concerning equiprobability modulo m generalize known [8] results on permutation polynomials to much wider classes of functions. The study was motivated by the observation that application of equiprobable modulo m functions as output functions to uniformly distributed in $\mathbb{Z}/M$ periodic sequences with period length M leads to new uniformly distributed in $\mathbb{Z}/N$ (with $N|M$) sequences of the same

period length M. In other words, each element of $\mathbb{Z}/N$ occurs at the period of such sequence the same number of times (but not necessarily once). Hashing with equiprobable modulo m functions the sequences, generated by already introduced methods, seems to be useful to design secure stream ciphers. Yet this will be an issue of the forthcoming paper and is out of the scope of the present one.

Note that proofs of our basic assertions use p-adic techniques. The problems stated above are restated in these terms. Actually the paper studies functions that are defined on (and take values in) the space $\mathbb{Z}_p$ of all p-adic integers, and that are ergodic with respect to the Haar measure (as well as those preserving this measure or equiprobable with respect to it), in the class of non-expanding functions, that is, those satisfying the Lipschitz condition with coefficient 1. From this view the results of the paper could be of interest for the theory of non-Archimedean dynamical systems: A number of statements could be easily interpreted as descriptions of ergodic dynamical systems in a phase space $\mathbb{Z}_p$.

The paper continues studies started in [11]: Here we prove some results announced in [11, 12, 14, 17] and establish new ones. Moving towards exact statements, for reader's convenience we recall some facts from the p-adic analysis and the theory of uniform distribution of sequences, following [6], [3] and [2]; we recall some necessary results, definitions and notation from [11] as well.

Here and after let p be a prime number. Consider a canonical representation $z = z_0 + z_1 p + z_2 p^2 + \dots$ of a p-adic integer $z \neq 0$, where $z_j \in \{0, 1, \dots, p-1\}$ $(j = 0, 1, 2, \dots)$; we denote $\operatorname{ord}_p z = \min\{j : z_j \neq 0\}$ the exponent of a maximal power of p which is a factor of z. By the definition, $\|z\|_p = p^{-\operatorname{ord}_p z}$ is a p-adic norm of z, $\|0\|_p = 0$. The valuation $\| \ \|_p$ could be expanded to the whole field $\mathbb{Q}_p$ of p-adic numbers (which is a quotient field of the ring $\mathbb{Z}_p$ of p-adic integers) in a standard way; so this valuation induces on $\mathbb{Q}_p$ a distance $d_p(u, v) = \|u - v\|_p$, with $\mathbb{Q}_p$ being a completion of the space $\mathbb{Q}$ of all rationals with respect to this distance. Note that often they use another terminology, where a distance is called a *metric*, a p-adic norm is called a *p-adic value*, whereas the term 'p-adic valuation' is reserved for ord_p. However, throughout the paper we mainly use the terminology of [3], with the only exception, speaking of 'p-adic norms' instead of 'p-adic values'.

The ring $\mathbb{Z}_p = \{u \in \mathbb{Q}_p : \|u\|_p \leq 1\}$ is compact in the space $\mathbb{Q}_p$, being a closure of the set $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Hence, $\mathbb{Z}_p$ is a separable compact metric space. The set of all cosets with respect to all ideals of the ring $\mathbb{Z}_p$ forms a base of the corresponding topology. Each coset $a + p^k\mathbb{Z}_p$ $(a \in \mathbb{Z}_p,\ k = 0, 1, 2, \dots)$ is an open (and simultaneously closed) ball of radius p^{-k}.

There exists a natural measure μ on $\mathbb{Z}_p$: putting $\mu(a + p^k\mathbb{Z}_p) = p^{-k}$, we then expand μ to the correponding σ-ring generated by all compact subsets of $\mathbb{Z}_p$ (these compact subsets are exactly all closed subsets of $\mathbb{Z}_p$). So we define uniquelly a measure on $\mathbb{Z}_p$, which is non-negative σ-additive regular normalized Borel and Haar measure in this case. Thus, μ is a natural probability measure on $\mathbb{Z}_p$. The probability measure on n-dimensional space $\mathbb{Z}_p^{(n)}$ could be defined in a similar way as a corresponding normalized Haar measure.

Now let $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ be a function that preserves all congruences of the ring $\mathbb{Z}_p$, i.e., $a\theta b$ implies $f(a)\theta f(b)$ for each congruence θ and all $a, b \in \mathbb{Z}_p$. Since each nontrivial congruence of the ring $\mathbb{Z}_p$ is an equivalence modulo some ideal $p^k\mathbb{Z}_p$

(we denote this congruence via $\cdot \equiv \cdot \pmod{p^k}$), it can be easily shown that the function f preserves all the congruences of the ring $\mathbb{Z}_p$ iff it satifies the Lipschitz condition with coefficient 1: $\|f(x)-f(y)\|_p \le \|x-y\|_p$. A function that preserves all congruences of a universal algebra is called *compatible*; we will use this term instead of the term 'conservative' of [11], since the latter in numerous papers on algebraic systems has attained another meaning, see [8, p. 45].

The class of all compatible functions on $\mathbb{Z}_p$ is rather wide: It contains all functions represented by polynomials with rational integer or p-adic integer coefficients, integer-valued analytic on $\mathbb{Z}_p$ functions, as well as integer-valued and meromorphic (in particular, rational) on $\mathbb{Z}_p$ functions with denominators equivalent to 0 modulo p at no point of $\mathbb{Z}_p$. Some other examples will be introduced further in the paper.

Recall that a function that is defined on some field F and takes values in F is called *integer-valued* iff all its values are integers of F whenever its arguments are integers of F. Further we study integer-valued functions on the field $\mathbb{Q}_p$; hence they map $\mathbb{Z}_p$ into $\mathbb{Z}_p$. In particular, we study integer-valued functions on $\mathbb{Q}$. A polynomial over a field F is called integer-valued iff it induces an integer-valued function on F. Note that integer-valued function f on F defines on the ring Z of all integers of F a function $f|_Z : Z \to Z$, which is not necessarily compatible on Z, i.e., does not necessarily preserve all congruences of Z; yet, if $f|_Z$ is compatible as a function on Z, then (in cases which do not lead to misunderstanding) we also call f compatible. Moreover, if a compatible integer-valued function f could be defined by a polynomial over F, we call compatible the corresponding polynomial too.

Note that the notion of compatible integer-valued function could be naturally expanded to a multivariate case — a valuation (and hence, a distance) on the space $\mathbb{Z}_p$ induces a (pseudo)-valuation (hence, a distance) on the n-dimensional space $\mathbb{Z}_p^{(n)}$ in a standard way: For $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{Z}_p^{(n)}$ we assume $\|\mathbf{u}\|_p = \max\{\|u_i\|_p : i = 1, 2, \dots, n\}$. So, the function

$$F = (f_1, \dots, f_m)\colon \mathbb{Z}_p^{(n)} \to \mathbb{Z}_p^{(m)}$$

is compatible iff it satisfies the Lipschitz condition with coefficient 1. In particular, all compatible on $\mathbb{Z}_p$ functions are continuous as functions of p-adic variables.

This obvious conclusion is important for applications. Each machine word, i.e., a word of some finite length in the alphabet $\{0, 1\}$, could be treated as a base-2 expansion of a non-negative integer rational. Then all the above mentioned bitwise logical operations and machine operations could be naturally continued to the set $\mathbb{Z}_2$ of all 2-adic integers in their canonical representations. Moreover, the above mentioned arithmetic operations could be continued to $\mathbb{Z}_2$ either. It could be easily demonstrated that all these operations (to be more precise, their uniquelly defined continuations to $\mathbb{Z}_2$) and all their compositions are compatible (hence, continuous) integer-valued functions on $\mathbb{Z}_2$: For exponentiation $(y, z) \mapsto (1 + 2y)^z$, and, in particular, for the inversion $y \mapsto (1+2y)^{-1}$ see 4.11; for the rest the assertion follows immediately from the corresponding definitions. We note here that an m-step shift towards less significant bits (i.e., the operation $\lfloor \frac{\cdot}{2^m} \rfloor$ of 'integer division', that is, a division with a subsequent truncation of the fractional part of the quotient) is *not* compatible, yet continuous and integer-valued function on $\mathbb{Z}_2$ (hence the results of the paper remain valid for compositions including the latter operation either, whenever the entire composition is compatible).

These considerations give an opportunity to apply, while studing compositions of the above mentioned operations, certain methods of the non-Archimedean (p-adic) analysis. Certainly, these techniques could be applied only to problems that are stated in appropriate terms (measures, distances, limits, derivatives, etc.).

It turnes out that some properties of functions, which traditionally have been treated as discrete mathematics issues, could be re-stated in these terms. We have already introduced one of these properties, namely, compatibility. It worth a brief notice in this connection that the so-called 'determined functions on superwords' of automata theory (which are functions defined on infinite sequences of $\{0,1\}$) after natural identification of superwords with elements of $\mathbb{Z}_2$ could be considered as compatible functions on $\mathbb{Z}_2$.

There exist other properties that could be considered in such a way. We consider a property of a compatible function $F = (f_1, \dots, f_m)\colon {\mathbb{Z}_p}^{(n)} \to {\mathbb{Z}_p}^{(m)}$ to be *equiprobable modulo* p^k. The latter by definition means that the function F induces on the nth Cartesian power $(\mathbb{Z}/p^k)^{(n)}$ of $\mathbb{Z}/p^k$ an equiprobable function $\bar{F} = (\bar{f}_1, \dots, \bar{f}_m)\colon (\mathbb{Z}/p^k)^{(n)} \to (\mathbb{Z}/p^k)^{(m)}$, i.e., each point of ${(\mathbb{Z}/p^k)}^{(m)}$ has the same number of F-preimages in ${(\mathbb{Z}/p^k)}^{(n)}$. In particular, for $m = n$ equiprobable modulo p^k functions are exactly *bijective modulo* p^k functions. We consider also an important (especially for pseudorandom generation) property of a bijective modulo p^k function F to be *transitivite modulo* p^k, which means that F induces on ${(\mathbb{Z}/p^k)}^{(n)}$ a permutation with a single cycle. Note that while defining notions of equiprobability, bijectivity or transitivity of a function F modulo p^k, we have assumed the compatibility of F.

By definition, the value of the induced function $\bar{f}_i(x)$ in the ring $\mathbb{Z}/p^k$ (which we denote as $f_i(x) \bmod p^k$) is the least non-negative residue modulo p^k of $f_i(x)$, i.e., $f_i(x) \bmod p^k = \alpha \in \{0, 1, \dots, p^k - 1\}$, where $\|f_i(x) - \alpha\|_p \le p^{-k}$. In view of the compatibility of the function f_i, the value of the function $\bar{f}_i(x)$ does not depend on the choice of the representative x in a coset of the ring $\mathbb{Z}_p^{(n)}$ with respect to the ideal $(p^k\mathbb{Z}_p)^{(n)}$; hence, the function F defines on $(\mathbb{Z}/p^k)^{(n)}$ a function $F \bmod p^k = (f_1(x) \bmod p^k, \dots, f_m(x) \bmod p^k)$, which takes values in $(\mathbb{Z}/p^k)^{(m)}$. Throughout the paper the latter function is denoted via $F \bmod p^k$, or via $\bar{F}$, when it does not lead to misunderstanding.

Now recall some definitions of the theory of measurable functions (cf. [1]). Let S and T be spaces with nonnegative normalized measures μ and τ, respectively, and let $f\colon S \to T$ be a measurable function, i.e., each full f-preimage $f^{-1}(U)$ of $U \subseteq T$ is μ-measurable for each τ-measurable U.

We say that the function f is (μ, τ)-*proportional*, iff for each pair of τ-measurable subsets $U, V \subseteq T$ the equality $\tau(U) = \tau(V)$ implies the equality $\mu(f^{-1}(U)) = \mu(f^{-1}(V))$. In case both μ, τ are probabilistic measures (e.g., are properly normalized Haar measures), then f is called (μ, τ)-*equiprobable* (or *equiprobable with respect to* μ and τ) iff $\mu(f^{-1}(U)) = \tau(U)$ for each τ-measurable $U \subseteq T$. For $S = T$ and $\mu = \tau$ we say that f *preserves measure* μ, iff $\mu(f^{-1}(U)) = \mu(U)$ holds for each μ-measurable U. Finally, if f preserves measure μ, and for μ-measurable subset U the equality $f^{-1}(U) = U$ implies that either $\mu(U) = 0$, or $\mu(U) = 1$, we say that f is μ-*ergodic* (or *ergodic with respect to* μ).

Note that in metric theory instead of terms 'measure-preserving function' or

'equiprobable function' they often use terms 'metric endomorhism' and 'metric homomorphism', and in dynamical systems theory they sometimes speak about 'metric transitivity' instead of ergodicity. Since throughout the paper we deal with the only measure, the properly normalized Haar measure, we omit mentioning this measure, so preserving the Haar measure, equiprobable (accordingly, ergodic) with respect to the Haar measure functions are referred as *measure-preserving*, *equiprobable* (or, accordingly, *ergodic*).

The following theorem holds:

1.1 Theorem. *A compatible function* $F\colon {\mathbb{Z}_p}^{(n)} \to {\mathbb{Z}_p}^{(m)}$ *is equiprobable* (*respectively, measure-preserving or ergodic*) *iff it is equiprbable* (*respectively, is bijective or transitive*) *modulo* p^k *for all* $k = 1, 2, \dots$. *A compatible and measure-preserving function* F *is bijective* (*consequently, is a metric automorphism*)*; moreover, it is an isometry of the space* $\mathbb{Z}_p^{(n)}$.

Note that further throughout the paper while proving ergodicity (equiprobability) of a compatible function with respect to the Haar measure we actually prove its transitivity (equiprobability) each modulo p^k, $k = 1, 2, \dots$, i.e., directly establish the properties we are interested in view of the problems mentioned above. That is why we omit the proof of this theorem 1.1: it is not related directly to the aims of this paper. Nevertheless throughout the paper we use the relevant terminology (e.g., we commonly speak of 'ergodicity' instead of 'transitivity modulo p^k for all $k = 1, 2, \dots$', etc.)

In connection with theorem 1.1 it is worth noticing, however, that the results of the paper related to description of measure-preserving or ergodic functions may be treated as description of non-Archimedean (i.e., ultrametric) dynamical systems $(\mathbb{Z}_p^{(n)}, F)$ with phase space $\mathbb{Z}_p^{(n)}$, discrete time, and with nonexpanding F (i.e. for each pair of points $\mathbf{a}, \mathbf{b}$ a distance between their F-images $F(\mathbf{a})$ and $F(\mathbf{b})$ does not exceed a distance between these points). In this sence theorem 2.2, for instance, might be condidered as a complete description (in terms of explicit formulae) of ergodic dynamical systems of the above mentioned kind when $p = 2$ and $n = 1$; together with theorem 3.11 it gives full description of twice integer-valued (i.e., having everywhere integer-valued derivative) ergodic dynamical systems. These themes, however, are not covered by this paper and will be considered in a forthcoming one.

Returning to the leading theme of the paper we note that for a wide class of compatible functions that are in some (properly defined in section 3) sence generalizations of uniformly differentiable on $\mathbb{Z}_p$ functions the bijectivity modulo p^k of a function for a *certain* k is equivalent to the property of being measure-preserving; the latter is equivalent to its bijectivity modulo p^k *for all* $k = 1, 2, 3, \dots$. The property of being transitive modulo p^k for a *certain* k turned out to be equivalent to the ergodicity of a function; the latter implies that the function is transitive modulo p^k *for all* $k = 1, 2, 3, \dots$. Finally, the equiprobability of a function modulo p^k for a *certain* k implies its equiprobability with respect to the Haar measure; the latter property is equivalent to equiprobability modulo p^k *for all* $k = 1, 2, 3 \dots$. The results of this kind are proved in section 3.

These results demonstrate the same remarkable effect originally enlighted by the Hensel's lemma; mamely, Hensel lifting, that is, a situation when a behavior

of a function modulo p^{k_0} for a certain k_0 controls its behavior modulo p^k for all $k = k_0 + 1, k_0 + 2, \dots$ and on the whole space $\mathbb{Z}_p$. This effect have been already observed while studying transitivity of some transformations. For instance, the necessary and sufficient conditions for the polynomial $f(x) = a + bx$ with integer rational a, b (see e.g., [2; 3.2.1.2, theorem A]) could be re-stated as follows: A polynomial $a + bx$ is transitive modulo p^k for some (that is, for all) $k \geq 2$ iff it is transitive modulo p for odd p or, respectively, modulo p^2 for $p = 2$. The general criterion for the transitivity modulo p^k of the polynomial f of arbitrary degree over integer rationals [15] demonstrates this effect either: For $p \neq 2, 3$ a polynomial f is transitive modulo p^k, $k \geq 3$, iff it is transitive modulo p^2; respectively, for $p = 2$ or $p = 3$ — iff it is transitive modulo p^3. Note by the way that the latter result holds for a much wider class of functions, even not necessarily analytic (see 4.9–4.10).

The results of section 3 show that a Hensel lifting of such properties as bijectivity or transitivity modulo p^k is a consequence of a specific character of p-adic distance and holds for various rather wide classes of functions. The values of k_0 from which the lifting starts are estimated in section 4.

The results of this kind are useful if for the given f one has to establish whether it has some property (e.g., transitivity or bijectivity) modulo p^k for a definite rather large k, for which direct verification is not possible. However, if one needs to construct out of prescribed operations a certain function that must be transitive (or bijective) modulo p^k, then explicit formulae are more convenient. Such formulae for bijective modulo 2^k polynomials over $\mathbb{Z}$ were obtained in [13], for transitive modulo 2^k polynomials over $\mathbb{Z}$ — in [15]. Explicit formulae for ergodic or measure-preserving compatible functions (in particular, for compatible integer-valued polynomials over $\mathbb{Q}$) that are defined on (and take values in) $\mathbb{Z}_2$ were obtained in [11]. In the current paper we obtain explicit formulae for compatible ergodic (or measure-preserving) functions on $\mathbb{Z}_p$ for odd p — see the next section.

2. Explicit formulae

Recall (see [3]) that each function $f\colon \mathbb{N}_0 \to \mathbb{Z}_p$ (or, respectively, $f\colon \mathbb{N}_0 \to \mathbb{Z}$) admits one and only one representation in the form of so-called *interpolation series*

$$f(x) = \sum_{i=0}^{\infty} a_i \binom{x}{i}, \qquad (\diamondsuit)$$

where $\binom{x}{i} = \dfrac{x(x-1)\cdots(x-i+1)}{i!}$ for $i = 1, 2, \dots$, and $\binom{x}{0} = 1$; $a_i \in \mathbb{Z}_p$ (respectively, $a_i \in \mathbb{Z}$), $i = 0, 1, 2, \dots$.

If f is uniformly continuous on $\mathbb{N}_0$ with respect to p-adic distance, it can be uniquely expanded to a uniformly continuous function on $\mathbb{Z}_p$. Hence the interpolation series for f converges uniformly on $\mathbb{Z}_p$. The following is true: The series $f(x) = \sum_{i=0}^{\infty} a_i \binom{x}{i}$, $\quad (a_i \in \mathbb{Q}_p,\ i = 0, 1, 2, \dots)$ converges uniformly on $\mathbb{Z}_p$ iff $\lim\limits_{i \to \infty}\limits^{p} a_i = 0$, where $\lim\limits^{p}$ is a limit with respect to p-adic distance; hence uniformly convergent series defines a uniformly continuos function on $\mathbb{Z}_p$. The latter function is integer-valued iff $a_i \in \mathbb{Z}_p$ for all $i = 0, 1, 2, \dots$.

Further throughout this section we assume that the function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is uniformly continuous on $\mathbb{Z}_p$, and that it is represented by series ($\diamondsuit$). The following three criteria hold (see [11]):

2.1 Theorem. (See 4.3 of [11]; cf. [5]) *The function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is compatible iff*

$$a_i \equiv 0 \pmod{p^{\lfloor \log_p i \rfloor}}$$

for all $i = p,\ p+1,\ p+2, \dots$. (Here and after for a real α we denote $\lfloor \alpha \rfloor$ an integral part of α, i.e., the nearest to α integer rational not exceeding α.)

2.2 Theorem. (See [11, 4.5]) *The function $f\colon \mathbb{Z}_2 \to \mathbb{Z}_2$ is compatible and measure-preserving iff it could be represented as*

$$f(x) = c_0 + x + \sum_{i=1}^{\infty} c_i\, 2^{\lfloor \log_2 i \rfloor + 1} \binom{x}{i},$$

where $c_0, c_1, c_2 \ldots \in \mathbb{Z}_2$.

2.3 Theorem. (See 4.7 of [11]) *The function $f\colon \mathbb{Z}_2 \to \mathbb{Z}_2$ is compatible and ergodic iff it could be represented as*

$$f(x) = 1 + x + \sum_{i=0}^{\infty} c_i\, 2^{\lfloor \log_2 (i+1) \rfloor + 1} \binom{x}{i},$$

where $c_0, c_1, c_2 \ldots \in \mathbb{Z}_2$.

For an arbitrary prime p condtions of theorems 2.2 and 2.3 are not necessary, yet sufficient. Namely, in this section we prove the following:

2.4 Theorem. *Let p be an odd prime. The function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$, which is represented in the form ($\diamondsuit$), is compatible and measure-preserving if the following congruences hold simultaneously:*

$$\begin{gathered} a_1 \not\equiv 0 \pmod{p}; \\ a_i \equiv 0 \pmod{p^{\lfloor \log_p i \rfloor + 1}}, \quad (i = 2, 3, \dots). \end{gathered}$$

The function f if compatible and ergodic if the following congruences hold simultaneously:

$$\begin{gathered} a_0 \not\equiv 0 \pmod{p}; \\ a_1 \equiv 1 \pmod{p}; \\ a_i \equiv 0 \pmod{p^{\lfloor \log_p (i+1) \rfloor + 1}}, \quad (i = 2, 3, \dots). \end{gathered}$$

To prove the theorem we need two additional results, which are of interest by their own.

2.5 Lemma. *Let p be an arbitrary prime, let $v\colon \mathbb{Z}_p \to \mathbb{Z}_p$ be a compatible function, and let c, d be p-adic integers, $c \not\equiv 0 \pmod p$. Then the function $g(x) = d + cx + pv(x)$ preserves measure, and the function $h(x) = c + x + p\Delta v(x)$ is ergodic.* (Here and after Δ is a difference operator: $\Delta v(x) = v(x+1) - v(x)$. Note that both g and h are obviously compatible since they are compositions of compatible functions.)

Proof of lemma 2.5. To start with, by induction on l we show that g is bijective modulo p^l for all $l = 1, 2, 3, \dots$. The assumption is obviously true for $l = 1$.

Assume it is true for $l = 1, 2, \dots, k-1$. Prove that it holds for $l = k$ either. Let $g(a) \equiv g(b) \pmod{p^k}$ for some p-adic integers a, b. Then $a \equiv b \pmod{p^{k-1}}$ by the induction hypothesis. Hence $pv(a) \equiv pv(b) \pmod{p^k}$ since v is compatible. Further, the congruence $g(a) \equiv g(b) \pmod{p^k}$ implies that $ca + pv(a) \equiv cb + pv(b) \pmod{p^k}$, and consequently, $ca \equiv cb \pmod{p^k}$. Since $c \not\equiv 0 \pmod p$, the latter congruence implies that $a \equiv b \pmod{p^k}$, proving the first assertion of the lemma.

To prove the rest part of the statement we note that the just proven assertion implies that h preserves measure. To prove the transitivity of h modulo p^k for all $k = 1, 2, 3, \dots$ we apply induction on k once again.

It is obvious that h is transitive modulo p. Assume that h is transitive modulo p^{k-1}. Then, since h induces a permutation on $\mathbb{Z}/p^k$ and since it is a compatible function, we conclude that the length of each cycle of this permutation must be a multiple of p^{k-1}. So to prove this permutation is single cycle it is sufficient to prove that the function

$$h^{p^{k-1}}(x) = \underbrace{h(h \dots (h}_{p^{k-1}}(x)) \dots)$$

induces a single cycle permuation on the ideal (p^{k-1}), generated by the element p^{k-1} of the ring $\mathbb{Z}/p^k$. In other words, it is sufficient to demonstrate that the function $\frac{1}{p^{k-1}} h^{p^{k-1}}(p^{k-1}x)$ is transitive modulo p.

Applying obvious direct calculations, we successively obtain that

$$h^1(x) = c + x + pv(x+1) - pv(x),$$

$$\dots \qquad \dots \qquad \dots$$

$$h^j(x) = h(h^{j-1}(x)) = cj + h^{j-1}(x) + pv(h^{j-1}(x)+1) - pv(h^{j-1}(x))$$
$$= cj + x + p\sum_{i=0}^{j-1} v(h^i(x)+1) - p\sum_{i=0}^{j-1} v(h^i(x)),$$

and henceforth. We recall that $h^0(x) = x$ by definition. So,

$$h^{p^{k-1}}(x) = cp^{k-1} + x + p\sum_{i=0}^{p^{k-1}-1} v(h^i(x)+1) - p\sum_{i=0}^{p^{k-1}-1} v(h^i(x)). \qquad (1)$$

Since h is transitive modulo p^{k-1} and compatible, we get now that

$$\sum_{i=0}^{p^{k-1}-1} v(h^i(x)+1) \equiv \sum_{i=0}^{p^{k-1}-1} v(h^i(x)) \equiv \sum_{z=0}^{p^{k-1}-1} v(z) \pmod{p^{k-1}},$$

and (1) implies then $h^{p^{k-1}}(x) \equiv cp^{k-1} + x \pmod{p^k}$. But $c \not\equiv 0 \pmod{p}$, so we conclude that the function $cp^{k-1} + x$ induces on the ideal (p^{k-1}) a single cycle permutation, thus proving the lemma. $\square$

2.6 Corollary. *Under assumptions of lemma 2.5 let p be an odd prime, and let $r \equiv 1 (\mathrm{mod} p)$. Then the function $c + rx + p\Delta v(x)$ is compatible and ergodic.*

Proof of the collorary 2.6. We have that $r = 1 + ps$ for a suitable $s \in \mathbb{Z}_p$. Now, since p is odd, the function $s\binom{x}{2}$ is compatible; consequently, the function $v_1(x) = s\binom{x}{2} + v(x)$ is compatible either. Yet $\Delta v_1(x) = sx + \Delta v(x)$, and it is sufficient now to apply lemma 2.5 to finish the proof of the corollary. $\square$

Proof of the theorem 2.4. Note that according to 2.1 a compatible function $v(x)$ could be represented as

$$v(x) = a + \sum_{i=1}^{\infty} b_i p^{\lfloor \log_p i \rfloor} \binom{x}{i},$$

where $a, b_1, b_2, \ldots \in \mathbb{Z}_p$. As $\lfloor \log_p i \rfloor = \lfloor \log_p (i+1) \rfloor$ for all $i = 1, 2, \ldots$ with the exception of $i = p^t - 1$, $(t = 1, 2, 3, \ldots)$, and as

$$\Delta v(x) = \sum_{i=1}^{\infty} b_i p^{\lfloor \log_p i \rfloor} \binom{x}{i-1}, \tag{1}$$

we finish the proof of the theorem, applying 2.5 and 2.6. $\square$

For $p = 2$ the above results imply one more useful criterion of ergodicity (or measure-preservation) of a function.

2.7 Theorem. *The function $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ is compatible and measure-preserving (respectively, is compatible and ergodic) iff it can be represented in the form $f(x) = c + x + 2v(x)$ (respectively, in the form $f(x) = 1 + x + 2\Delta v(x)$), where $c \in \mathbb{Z}_2$ and $v(x)$ is a compatible function.*

Proof. Follows easily from 2.1–2.3 and 2.5 in combination with (1) of the proof of the theorem 2.4. $\square$

Both 2.5–2.6 and theorem 2.7 could be applied in order to consruct measure-preserving or ergodic functions as compositios of the given compatible functions. For instance, putting $v(x) = (x^2) \,\mathrm{XOR}(x + 32 \,\mathrm{AND}\, x)$ (this function is compatible as a composition of compatible functions) we conclude that the function

$$7 + x + 2((x^2 + 2x + 1) \,\mathrm{XOR}(x + 1 + 32 \,\mathrm{AND}(x + 1))) - 2(x^2 \,\mathrm{XOR}(x + 32 \,\mathrm{AND}\, x))$$

is ergodic. This conclusion is not very easy to verify by direct application of theorems 2.2 or 2.3.

By the way, for $p = 2$ the statement of theorem 2.7 could be slightly modified to make it a little bit more convenient for the construction of ergodic functions out of addition and bitwise logical operations (like bitwise exclusive 'or', XOR, bitwise 'and', AND, or bitwise negation NEG). Namely, it could be easily seen that in the ring $\mathbb{Z}_2$ there holds an identity $z + \mathrm{NEG}(z) = -1$. Hence, $\Delta v(x) = v(x+1) - v(x) = v(x+1) + \mathrm{NEG}(v(x)) + 1$, and we obtain the following

2.8 Proposition. *A function $f\colon \mathbb{Z}_2 \to \mathbb{Z}_2$ is compatible and ergodic iff it can be represented in one (hence, all) of the following forms:*

$$f(x) = 1 + x + 2(v(x+1) + \mathrm{NEG}\, v(x)),$$
$$f(x) = 2 + x + 2v(x+1) + \mathrm{NEG}(2v(x)),$$
$$f(x) = 3 + x + 2v(x+1) + 2\mathrm{NEG}v(x),$$

for a suitable compatible function $v\colon \mathbb{Z}_2 \to \mathbb{Z}_2$.

Since multiplication by 2 is just a 1-digit shift of a base-2 expansion of a number towards more significant bits, the proposition 2.8 could be applied in order to construct pseudorandom number generators out of the 'fast' computer commands, like addition, bitwise logical operations, and shifts towards more significant bits, by implementing the function v as a composition of them.

It worth noticing also that all the functions described in 2.4 – 2.8 are 'affine modulo p', i.e., induce on $\mathbb{Z}/p$ a transformation of the form $x \mapsto a + bx$.

3. Hensel lifting.

This section studies conditions when a function, which belongs to an important class of uniformly differentiable modulo p^k functions (the latter being defined below), is equiprobable, measure-preserving or ergodic. As a rule, the results of the section demonstrate the effect of Hensel lifting, which was mentioned in the introduction: Speaking loosely, if a function F has some property modulo p^{k_0} then it has this property modulo p^n for all $n \geq k_0$. Besides, it worth noticing here that the results of this section, contrasting those of the previous one, provide some tools to construct measure-preserving or ergodic functions that are not necessarily affine modulo p. In fact, certain techniques based on the ideas of this section could be developed; these techniques enables one 'to lift' an arbitrary transitive transformation of the ring $\mathbb{Z}/p^{k_0}$ to the function on $\mathbb{Z}_p$, which is transitive modulo p^k for all $k = k_0, k_0+1, k_0+2, \ldots$. This is the main reason we introduce a notion of asymptotically compatible function below. However, the techniques themselves are not discussed here being left to a forthcoming paper.

Firstly, recall some generalizations of our basic notions (see 5.1 of [11]).

3.1 Definition. Let $F = (f_1, \ldots, f_m)\colon \mathbb{Z}_p^{(n)} \to \mathbb{Z}_p^{(m)}$ be a function, which is not necessarily compatible. The function F is called (*asymptotically*) *equiprobable*, iff for all $k = 1, 2, \ldots$ (respectively, for all sufficiently large $k \in \mathbb{N}$) it is *equiprobable modulo* p^k, that is, the restriction $F \bmod p^k = (f_1 \bmod p^k, \ldots, f_m \bmod p^k)$ of the function F to the set $\{0, 1, \ldots, p^k - 1\}^{(n)}$ is an equiprobable function. (Note that in cases which do not lead to misunderstanding we identify the set $\{0, 1, \ldots, p^k - 1\}^{(n)}$ with the set of all elements of the ring $(\mathbb{Z}/p^k)^{(n)}$). By the analogy, we say that F is *asymptotically measure-preserving* (respectively, that F is *asymptotically ergodic*), iff $F \bmod p^k$ is a bijective (respectively, transitive) transformation of the ring $(\mathbb{Z}/p^k)^{(n)}$ for all sufficiently large k. Lastly, we say that F is *asymptotically compatible* iff there exists positive integer rational N such that for all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_p^{(n)}$ and all $k \geq N$ a congruence $\mathbf{a} \equiv \mathbf{b} \pmod{p^k}$ implies a congruence $F(\mathbf{a}) \equiv F(\mathbf{b}) \pmod{p^k}$.

By definition, for $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ of $\mathbb{Q}_p^{(n)}$ the congruence $\mathbf{a} \equiv \mathbf{b} \pmod{p^s}$ means that $\|a_i - b_i\|_p \le p^{-s}$ (or, the same, that $a_i = b_i + c_i p^s$ for suitable $c_i \in \mathbb{Z}_p$, $i = 1, 2, \dots, s$); that is $\|\mathbf{a} - \mathbf{b}\|_p \le p^{-s}$. In other words, a function is asymptotically compatible iff for some $N \in \mathbb{N}_0$ it satisfies the Lipschitz condition with coefficient 1 for each pair of points that are at least as close one to another as p^{-N}. Since $\mathbb{Z}_p^{(n)}$ is compact, F is asymptotically compatible iff it satisfies the Lipschitz condition with coefficient 1 locally.

Now for reader's convenience we recall some facts of [11]. A function $F = (f_1, \dots, f_m)\colon \mathbb{Z}_p^{(n)} \to \mathbb{Z}_p^{(m)}$ is called *differentiable modulo* p^k at the point $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{Z}_p^{(n)}$ iff there exist a positive integer rational N and an $n \times m$ matrix $F'_k(\mathbf{u})$ over $\mathbb{Q}_p$ (which is called *the Jacobi matrix modulo* p^k of the function F at the point $\mathbf{u}$) such that for each positive integer rational $K \ge N$ and each $\mathbf{h} = (h_1, \dots, h_n) \in \mathbb{Z}_p^{(n)}$ the inequality $\|\mathbf{h}\|_p \le p^{-K}$ implies a congruence

$$F(\mathbf{u} + \mathbf{h}) \equiv F(\mathbf{u}) + \mathbf{h} \cdot F'_k(\mathbf{u}) \pmod{p^{k+K}}. \qquad (\heartsuit)$$

In the case $m = 1$ the Jacobi matrix modulo p^k is called a *differential modulo* p^k. In case $m = n$ a determinant of the Jacobi matrix modulo p^k is called the *Jacobian modulo* p^k. The entries of the Jacobi matrix modulo p^k are called *partial derivatives modulo* p^k of the function F at the point $\mathbf{u}$. A partial derivative (respectively, a differential) modulo p^k are sometimes denoted as $\frac{\partial_k f_i(\mathbf{u})}{\partial_k x_j}$ (respectively, as $d_k F(\mathbf{u}) = \sum_{i=1}^n \frac{\partial_k F(\mathbf{u})}{\partial_k x_i} d_k x_i$).

The definition immediately implies that partial derivatives modulo p^k of the function F are defined up to a p-adic integer summand with a p-adic norm not exceeding p^{-k}. In those cases when all partial derivatives modulo p^k at all points of $\mathbb{Z}_p^{(n)}$ are p-adic integers we say that the function F has *integer-valued derivative modulo* p^k; in these cases we associate to each partial derivative modulo p^k a unique element of the ring $\mathbb{Z}/p^k$, and the Jacobi matrix modulo p^k at each point $\mathbf{u} \in \mathbb{Z}_p^{(n)}$ thus can be considered as a matrix over a ring $\mathbb{Z}/p^k$.

Under the latter agreement the 'rules of derivation modulo p^k' are similar to those of classical analysis. The only difference is that they are congruences modulo p^k and not equalities. For instance, if both functions $G\colon \mathbb{Z}_p^{(s)} \to \mathbb{Z}_p^{(n)}$ and $F\colon \mathbb{Z}_p^{(n)} \to \mathbb{Z}_p^{(m)}$ are differentiable modulo p^k at the points, respectively, $\mathbf{v} = (v_1, \dots, v_s)$ and $\mathbf{u} = G(\mathbf{v})$, and their partial derivatives modulo p^k at these points are p-adic integers, then the composition $F \circ G\colon \mathbb{Z}_p^{(s)} \to \mathbb{Z}_p^{(m)}$ of these functions is uniformly differentiable modulo p^k at the point $\mathbf{v}$, all its partial derivatives modulo p^k at this point are p-adic integers, and $(F \circ G)'_k(\mathbf{v}) \equiv G'_k(\mathbf{v}) F'_k(\mathbf{u}) \pmod{p^k}$.

By the analogy with a classical case we define for the function F a notion of *uniform differentiability modulo* p^k *on* $\mathbb{Z}_p^{(n)}$; the least $K \in \mathbb{N}$ such that $(\heartsuit)$ holds simultaneously for all $\mathbf{u} \in \mathbb{Z}_p^{(n)}$, whereas $\|h_i\|_p \le p^{-K}$, $(i = 1, 2, \dots, n)$, is denoted via $N_k(F)$. The latter number plays an important role in further coniderations.

We recall that accordingly to 2.12 of [11] all partial derivatives modulo p^k of the uniformly differentiable modulo p^k function F are periodic functions with a period $p^{N_k(F)}$. This in particular implies that each partial derivative modulo p^k can be considered as a function defined on $\mathbb{Z}/p^{N_k(F)}$. Moreover, a function $F =$

$(f_1, \dots, f_m)\colon \mathbb{N}_0^{(n)} \to \mathbb{N}_0^{(m)}$ together with all its (partial) derivatives modulo p^k could be expanded on the whole space $\mathbb{Z}_p^{(n)}$ since periodic functions with a period p^N are uniformly differentiable on $\mathbb{Z}_p$, and their derivatives vanish everywhere on $\mathbb{Z}_p$.

Here and after in this section let $F = (f_1, \dots, f_m)\colon \mathbb{Z}_p^{(n)} \to \mathbb{Z}_p^{(m)}$ and $f\colon \mathbb{Z}_p^{(n)} \to \mathbb{Z}_p$ be functions that are uniformly differentiable on $\mathbb{Z}_p^{(n)}$ modulo p. This is relatively weak restriction since all uniformly differentiable on $\mathbb{Z}_p^{(n)}$ functions, as well as the functions that are uniformly differentiable on $\mathbb{Z}_p^{(n)}$ modulo p^k for some $k \geq 1$, are uniformly differentiable on $\mathbb{Z}_p^{(n)}$ modulo p.

Examples of functions that are not uniformly differentiable on $\mathbb{Z}_p^{(n)}$, yet are uniformly differentiable on $\mathbb{Z}_p^{(n)}$ modulo p, are the function $f(x, y) = x \operatorname{XOR} y$ for $p = 2$ and its corresponding analogons for $p \neq 2$; all partial derivatives modulo p of these functions are congruent to 1 modulo p at all points (see [11]). Note, by the way, that the already introduced function $\bmod p^n\colon \mathbb{Z}_p \to \mathbb{Z}/p^n$, the 'reduction modulo p^n', is uniformly differentiable on $\mathbb{Z}_p$ (its derivative is 0 at all points). The function $f(x, y) = x \operatorname{AND} y$ is differentiable modulo 2 at no point of $\mathbb{Z}_2^{(2)}$, yet it is uniformly differentiable with respect to x for each $y \in \mathbb{Z}$; its derivative is 0 for $y \geq 0$, and it is 1 in the opposite case.

It turnes out that properties of being asymptotically compatible or asymptotically measure-preserving impose certain restrictions on p-adic norms of derivatives modulo p of the given function.

3.2 Proposition. *If the function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ asymptotically preserves measure, then $\|f_1'(u)\|_p \geqslant 1$ at all points $u \in \mathbb{Z}_p$.*

Proof. Since a derivative modulo p^k of the function f is periodic with period $p^{N_k(f)}$, it is sufficient to prove the proposition assuming $u \in \mathbb{N}_0$. The definition of differentiability modulo p^k implies that for $K \geq N_1(f)$ and for $u \in \mathbb{N}_0$ the congruence

$$f(u + h) \equiv f(u) + hf_1'(u) \pmod{p^{K+1}} \tag{1}$$

holds whenever $\|h\|_p \leq p^{-K}$. Assuming $\|f_1'(u)\|_p < 1$ for some $u \in \mathbb{N}_0$, the condition $f_1'(u) \equiv 0 \pmod p$ together with congruence (1) imply that $f(u + p^K) \equiv f(u) \pmod{p^{K+1}}$. The latter congruence means that for all $K \geq N_1(f)$, such that $u + p^K \leq p^{K+1} - 1$, the function f is not bijective modulo p^{K+1}. A contradiction. □

3.3 Corollary. *If under assumptions of 3.2 the function f is uniformly differentiable, then $\|f'(u)\|_p \geq 1$ for all $u \in \mathbb{Z}_p$.*

Proof. The definition of a derivative modulo p immediately implies that

$$f_1'(u) \equiv f'(u) \pmod p$$

for all $u \in \mathbb{Z}_p$. Thus $f'(u) = f_1'(u) + ps(u)$ for a suitable function $s\colon \mathbb{Z}_p \to \mathbb{Z}_p$. Yet if $\|f_1'(u)\|_p \geq 1$, then the latter equality obviously implies that $\|f'(u)\|_p \geq 1$ by the properties of p-adic distance. Now the conclusion follows from 3.2. □

The inverse of 3.2 is not true: The function $\frac{x^2 - x}{2}$ on $\mathbb{Z}_2$ serves an obvious counterexample. It vanishes both at 0 and at 1, but the 2-adic norm of its derivative is 2 everywhere on $\mathbb{Z}_2$. Nevertheless, functions of this kind are locally injective. Namely, the following is true:

3.4 Proposition. *If the function* $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ *is uniformly differentiable modulo* p*, and if* $\|f_1'(u)\|_p \geq 1$*, then the space* $\mathbb{Z}_p$ *can be represented as a disjoint union of a finite number of open (and simultaneously closed) balls* U*, for which the following holds: If* $a, b \in U$*,* $k \geq N_1(f)$ *and* $a \not\equiv b \pmod{p^k}$*, then* $f(a) \not\equiv f(b) \pmod{p^k}$*.*

Proof. Consider a union

$$\mathbb{Z}_p = \bigcup_{a=0}^{p^N-1} (a + p^N \mathbb{Z}_p),$$

where $N = N_1(f)$. Each set $U = a + p^N\mathbb{Z}_p$ is an open (and at the same time closed) ball of radius p^{-N} (see [3]). Let $u, v \in U$, and let $u \neq v$. Then $v = u + h$, where $\|h\|_p = p^{-K}$ for a suitable positive integer rational $K \geq N$. The definition of differentiability modulo p implies that

$$f(u + h) \equiv f(u) + hf_1'(u) \pmod{p^{K+1}}. \tag{1}$$

Thus, if $f(u) \equiv f(v) \pmod{p^K}$, then (1) implies that $\|f_1'(u)\|_p = p^{-1} < 1$. A contradiction. □

Proposition 3.4 implies that if the p-adic norm of a uniformly differentiable modulo p function is not less then 1 everywhere on $\mathbb{Z}_p$, then this function might 'glue together modulo p^k' for a sufficiently large k only those points that lie in distinct balls of the statement of 3.4. From here it follows

3.5 Proposition. *Let the function* $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ *be uniformly differentiable modulo* p *on* $\mathbb{Z}_p$*. Then* f *asymptotically preserves measure iff the following condition hold simultaneously:*

(1) $\|f_1'(u)\|_p \geq 1$ *at all points* $u \in \mathbb{Z}_p$*;*
(2) $f(a) \not\equiv f(b) \pmod{p^n}$ *for all* $n, a, b \in \mathbb{N}_0$ *such that* $\|a - b\|_p \geq p^{-N_1(f)}$ $0 \leq a, b \leq p^n - 1$. □

A. A. Nechaev (private communication) noticed that the function $f(x) = \frac{x^2+x}{2}$ on $\mathbb{Z}_2$ asymptotically preserves measure (this also follows from 3.5). Thus, if a compatible function $g\colon \mathbb{Z}_2 \to \mathbb{Z}_2$ asymptotically preserves measure (all these functions are characterized in 2.2), then the function $h(x) = g(f(x))$ is uniformly differentiable modulo $p = 2$ and asymptotically preserves measure, and $\|g_1'(u)\|_2 = 2$ at all points $u \in \mathbb{Z}_2$. There are no other asymptoticaly measure-preserving functions $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ that are uniformly differentiable modulo p, and which derivatives modulo p have norms not less then 1 everywhere on $\mathbb{Z}_p$, [10]. The proof of the latter statement involves not only p-adic techniques, but algebraic geometry techniques as well.

The latter notice illustrates the fact that the second condition of the criterion 3.5 is rather difficult to verify since one has to calculate values of a function at infinite number of points. However, the problem might be simplified by imposing certain restrictions on the function. Namely, we will assume additionally that f maps each ball of radius p^{-M} (with $M \geq N_1(f)$) into a ball of radius p^{-M} (consequently, f is asymptotically compatible). This restriction is equvalent to the property of derivative modulo p to be integer-valued everywhere on $\mathbb{Z}_p$.

3.6 Proposition. *If for some $M \geq N_1(f)$ a uniformly differentiable modulo p function f maps each ball of radius p^{-M} into a ball of radius p^{-M}, then $f_1'(a) \in \mathbb{Z}_p$ for all $a \in \mathbb{Z}_p$. Vice versa, each uniformly differentiable modulo p function such that its derivative modulo p is integer-valued everywhere on $\mathbb{Z}_p$, maps each ball of radius p^{-M} into a ball of radius p^{-M}, for all $M \geq N_1(f)$.*

Proof. If $M \geq N_1(f)$ and $\|h\|_p \leq p^{-M}$, then the definition of uniform differentiability modulo p^k (see 2.4 of [11]) implies that

$$f(u+h) \equiv f(u) + hf_1'(u) \pmod{p^{M+1}} \tag{1}$$

for all $u \in \mathbb{Z}_p$. On the other hand, the inclusion $f(a + p^M\mathbb{Z}_p) \subseteq f(a) + p^M\mathbb{Z}_p$ implies that

$$\|f(u+h) - f(u)\|_p \leq p^{-M} \tag{2}$$

for all h with $\|h\|_p \leq p^{-M}$. Comparing (1) and (2) we see that $\|f_1'(u)\|_p \leq 1$. The inverse statement is equivalent to the asymptotic compatibility of f (see 2.10 of [11]). $\square$

Henceforth in the section we additionally assume that f and F have integer-valued derivatives modulo p. In particular, this implies that both f and F are asymptotically compatible (see 2.10 and 2.11 of [11]). Now we state necessary and sufficient conditions the function F must satisfy to be measure-preserving, and sufficient conditions to be equiprobable.

3.7 Theorem. *Let the function $F = (f_1, \dots, f_m)\colon \mathbb{Z}_p^{(n)} \to \mathbb{Z}_p^{(m)}$ be uniformly differentiable modulo p and let all its partial derivatives modulo p be integer-valued on $\mathbb{Z}_p$. Then F is asymptotically equiprobable if it is equiprobable modulo p^k for some $k \geq N_1(F)$ and rank of its Jacobi matrix $F_1'(u)$ modulo p is exactly m at all points $\mathbf{u} = (u_1, \dots, u_n) \in (\mathbb{Z}/p^k)^{(n)}$.*

Proof. For $\xi \in (\mathbb{Z}/p^s)^{(m)}$ denote $F_s^{-1}(\xi) = \{\gamma \in (\mathbb{Z}/p^s)^{(n)} : F(\gamma) \equiv \xi \pmod{p^s}\}$. Let $s \geq k \geq N_1(F)$. Since F is asymptotically compatible, and hence F is a sum of a compatible function and a periodic function with period $p^{N_1(F)}$ (see 2.10 of [11]), we conclude that if $\eta \in F_{s+1}^{-1}(\xi)$, then $\bar{\eta} \in F_s^{-1}(\bar{\xi})$. Here, in accordance with our agreement in the introduction, $\bar{\alpha} = (\bar{\alpha}_1, \dots, \bar{\alpha}_m) \in (\mathbb{Z}/p^s)^{(m)}$ stands for $\alpha \bmod p^s = (\alpha_1 \bmod p^s, \dots, \alpha_m \bmod p^s)$, where $\alpha = (\alpha_1, \dots, \alpha_m) \in (\mathbb{Z}/p^{s+1})^{(m)}$. Put $\lambda = \bar{\eta} + p^s\sigma \in (\mathbb{Z}/p^{s+1})^{(n)}$, where $\sigma \in (\mathbb{Z}/p)^{(n)}$. In view of the uniform differentiability of the function F modulo p (see ($\heartsuit$)), we have

$$F(\lambda) \equiv F(\eta) + p^s\sigma F_1'(\bar{\eta}) \pmod{p^{s+1}}. \tag{1}$$

Since $F(\bar{\eta}) \equiv \bar{\xi} + p^k\beta \pmod{p^{s+1}}$ and $\xi = \bar{\xi} + p^s\gamma$ for suitable $\beta, \gamma \in (\mathbb{Z}/p)^{(m)}$, in view of (1) we conclude that $\lambda \in F_{s+1}^{-1}(\xi)$ iff $\bar{\lambda} \in F_s^{-1}(\xi)$ (i.e., $\bar{\eta} \in F_s^{-1}(\xi)$) and α satisfies the following linear system over a field $\mathbb{Z}/p$:

$$\beta + \alpha F_1'(\bar{\eta}) = \gamma. \tag{2}$$

Thus, if columns of the matrix $F_1'(\bar{\eta})$ are linearly independent over $\mathbb{Z}/p$, then linear system (2) has exactly p^{n-m} pairwise distinct solutions for arbitrary $\beta, \gamma \in (\mathbb{Z}/p)^{(m)}$. From here it follows that

$$|F_{s+1}^{-1}(\xi)| = |F_s^{-1}(\xi)|p^{n-m}. \tag{3}$$

Hence, if F is equiprobable modulo p^k (i.e., if $|F_s^{-1}(\bar{\xi})|$ does not depend on $\bar{\xi}$) and if rank of the matrix $F_1'(\bar{\eta})$ is m, then (3) implies that F is equiprobable modulo p^{s+1}. $\square$

3.8 Corollaries. 1° *Under assumptions of theorem 3.7 let $m = 1$. Then F if asymptotically equiprobable if F is equiprobable modulo p^k for some $k \geq N_1(F)$, and the differential d_1F modulo p of the function F vanishes at no point of $(\mathbb{Z}/p^k)^{(n)}$.*

2° *Let $f(x_1, \dots, x_n)$ be a polynomial in variables $x_1, \dots, x_n$, and let all coefficients of f are p-adic integers. The polynomial f is equiprobable if it is equiprobable modulo p and all its partial derivatives vanishes simultaneously modulo p at no point of $(\mathbb{Z}/p)^{(n)}$* (i.e., are simultaneously congruent modulo p nowhere).

Proof. Assertion 1° trivially follows from 3.7. In turn, 2° immediately follows from 1°, since for all $f \in \mathbb{Z}[x_1, \dots, x_n]$ holds $N_1(f) \leq 1$. We need only to prove the latter inequality.

By Taylor's formula,

$$f(x_1 + h_1, \dots, x_n + h_n) = f(x_1, \dots, x_n) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i} + Q \tag{1}$$

where $Q \in \mathbb{Z}[x_1, \dots, x_n, h_1, \dots, h_n]$, and each monomial in a canonical representation of the polynomial Q is of degree not less then 2 with respect to variables $h_1, \dots, h_n$. Since $\|(h_1, \dots, h_n)\|_p = p^{-s}$, where $s \geq 1$, for all values of $x_1, \dots, x_n$ we have $Q \equiv 0 \pmod{p^{2s}}$. In view of (1) this proves the inequality. $\square$

For $m = n$ the above stated sufficient conditions of asymptotical equiprobability are also necessary ones.

3.9 Theorem. *A uniformly differentiable modulo p function*

$$F = (f_1, \dots, f_m) \colon \mathbb{Z}_p^{(n)} \to \mathbb{Z}_p^{(n)}$$

with integer-valued derivatives modulo p asymptotically preserves measure if and only if it is bijective modulo $p^{N_1(F)}$ and its Jacobian modulo p vanishes at no point of $(\mathbb{Z}/p^{N_1(F)})^{(n)}$ (an equivalent condition: Iff F is bijective modulo $p^{N_1(F)+1}$).

Proof. If F is bijective modulo $p^{N_1(F)}$, and if its Jacobian modulo p vanishes nowhere on $(\mathbb{Z}/p^{N_1(F)})^{(n)}$, then in view of 3.7 F is asymptotically equiprobable, hence, asymptotically preserves measure, since $m = n$.

Vise versa, let F asymptotically preserve measure, i.e., let F be bijective modulo p^k for all $k \geq N$, where N is some positive integer rational. Now take $k \geq \max\{N, N_1(F)\}$, then the definition of uniform differentiability modulo p implies that

$$F(u + p^k\alpha) \equiv F(u) + p^k \alpha F_1'(u) \pmod{p^{k+1}} \tag{1}$$

for all $u, \alpha \in \mathbb{Z}_p$. Here $F_1'(u)$ is an $n \times n$ matrix over a field $\mathbb{Z}/p$. If $\det F_1'(u) \equiv 0 \pmod{p}$ for some $u \in \mathbb{Z}_p^{(n)}$ (or, the same, for some $u \in \{0, 1, \dots, p^{N_1(F)} - 1\}^{(n)}$ in view of the periodicity of partial derivatives modulo p), then there exists $\alpha \in \{0, 1, \dots, p-1\}^{(n)}, \alpha \not\equiv (0, \dots, 0) \pmod{p}$, such that $\alpha F_1'(u) \equiv (0, \dots, 0) \pmod{p}$. But then (1) implies that $F(u + p^k\alpha) \equiv F(u) \pmod{p^{k+1}}$. The latter contradicts

the bijectivity modulo p^{k+1} of the function F, since for $u \in \{0, 1, \dots, p^{N_1(F)} - 1\}^{(n)}$ we have $u, u + p^k\alpha \in \{0, 1, \dots, p^{k+1} - 1\}^{(n)}$ and $u + p^k\alpha \neq u$.

Now we prove the criterion in the equivalent form. Let F be bijective modulo $p^{N_1(F)}$. Then assuming $k = N_1(F)$ in the above argument, we conclude that $\det F_1'(u) \not\equiv 0 \pmod p$ for all $u \in \mathbb{Z}_p^{(n)}$. According to 3.7, this implies that F asymtotically preserves measure.

Let F asymptotically preserve measure, and let it be not bijective modulo p^k for some $k \geq N_1(F)$. We prove that in this case F is not bijective modulo p^{k+1}.

Choose $u, v \in \{0, 1, \dots, p^k - 1\}^{(n)}$ such that $u \neq v \quad F(u) \equiv F(v) \pmod{p^k}$. Then either $F(u) \equiv F(v) \pmod{p^{k+1}}$ (i.e., F is not bijective modulo p^{k+1}), or $F(u) \not\equiv F(v) \pmod{p^{k+1}}$. Yet in the latter case we have $F(u) \equiv F(v) + p^k\alpha \pmod{p^{k+1}}$ for some $\alpha \in \{0, 1, \dots, p - 1\}^{(n)}$, $\alpha \not\equiv (0, \dots, 0) \pmod p$. Consider $u_1 = u + p^k\beta$, where $\beta \in \{0, 1, \dots, p - 1\}^{(n)}$ with $\beta \not\equiv (0, \dots, 0) \pmod p$ and $\beta F_1'(u) + \alpha \equiv (0, \dots, 0) \pmod p$. Such β exists, since F asymptotically preserves measure and, consequently, $\det F_1'(u) \not\equiv 0 \pmod p$, as it have been proven already. Now the definition of uniform differentiability modulo p implies that

$$F(u+p^k\beta) \equiv F(u)+p^k\beta F_1'(u) \equiv F(v)+p^k\alpha+p^k\beta F_1'(u) \equiv F(v) \pmod{p^{k+1}}, \quad (2)$$

where $u + p^k\beta \in \{0, 1, \dots, p^{k+1} - 1\}^{(n)}$ and $u + p^k\alpha \neq v$ (since $u \neq v$). Thus (2) in combination with our assumption imply that F is not bijective modulo p^{k+1}. Applying this argument sufficient number of times, we conclude that F is not bijective modulo p^s for all $s \geq k$. But at the same time F asymptotically preserves measure. A contradiction. $\square$

3.10 Corollaries. 1° *If $n = 1$ under the assumptions of theorem 3.9, then F asymptotically preserves measure iff it is bijective modulo $p^{N_1(F)}$ and its derivative modulo p vanishes at no point of $\{0, 1, \dots, p^{N_1(F)} - 1\}$.*

2° (cf. [8, Ch. 4, sections 4–5]) *Let $F = (f_1, \dots, f_m)\colon \mathbb{Z}_p^{(n)} \to \mathbb{Z}_p^{(n)}$, where $f_i(x_1, \dots, x_n) \in \mathbb{Z}_p[x_1, \dots, x_n]$, $i = 1, 2, \dots, n$. Then F preserves measure iff F is bijective modulo p and $\det F'(u) \not\equiv 0 \pmod p$ for all $u \in \{0, 1, \dots, p - 1\}^{(n)}$ (an equivalent statement: Iff F is bijective modulo p^2).*

3° *Let $A = \langle \mathbb{Z}_p; \Omega \rangle$ be a universal algebra of finite signature Ω, and let all operations of Ω are uniformly differentiable modulo p functions that have integer-valued derivatives modulo p. Then a polynomial over A defines an asymptotically measure-preserving function iff it is bijective modulo $p^{k(A)}$, where $k(A) = \max\{N_1(\omega) : \omega \in \Omega\} + 1$.*

Proof. Assertion 1° trivially follows from 3.9. Assertion 2° holds in view of 3.9, since $N_1(F) \leq 1$ (see proof of the corollary 3.8). A compositin $F \circ G$ of functions F and G, which are both uniformly differentiable modulo p and have integer-valued derivatives modulo p, is uniformly differentiable modulo p function that has an integer-valued derivative modulo p, and $N_1(F \circ G) \leq \max\{N_1(F), N_1(G)\}$. The latter proves assertion 3°. $\square$

Comparing statements 3.7 and 3.9 one may ask a natural question whether sufficient conditions of 3.7 are necessary. The answer is negative: Results of [9] make it possible to construct the following counterexample.

Consider a function $f(x,y) = 2x + y^3$ on $\mathbb{Z}_2$. As f is a polynomial over $\mathbb{Z}$, then it is uniformly differentiable, has integer-valued derivatives, and $df = 2dx + 3y^2dy$. So, $df \equiv 0 \pmod 2$ if $y \equiv 0 \pmod 2$. Nevertheless, f induces an equiprobable function $(\mathbb{Z}/2^n)^{(2)} \to \mathbb{Z}/2^n$ for every $n = 1, 2, \dots$. Here is a proof.

For $n = 1$ we have that $f(x,y) \equiv y \pmod 2$ is an an equiprobable function on $\mathbb{Z}/2$. Let $n > 1$. We will show that for every $z \in \mathbb{Z}/2^n$ there exist exactly 2^n pairs (x,y), such that $f(x,y) \equiv z \pmod{2^n}$ and $(x,y) \in \{0,1,\dots,2^n-1\}^{(2)}$.

In fact, if $z = 1 + 2r$ for some $r \in \{0,1,\dots,2^{n-1}-1\}$, then it follows that $y = 1 + 2k$ for some $k \in \{0,1,\dots,2^{n-1}-1\}$. So $2x + (1+2k)^3 \equiv 1 + 2r \pmod{2^n}$ implies $x + 3k + 6k^2 + 4k^3 \equiv r \pmod{2^{n-1}}$. The left hand part of the latter congruence is a polynomial $\phi(x,k)$ in x, k. It is equiprobable in view of $3.8, 2^\circ$, since $d\phi \equiv dx + dk \pmod 2$ (and hence this differential vanishes modulo 2 nowhere) and since the function $\phi \equiv x + k \pmod 2$ is obviously an equiprobable modulo 2. This implies that the congruence $\phi(x,k) \equiv r \pmod{2^{n-1}}$ in unknowns x, k has exactly 2^{n-1} solutions in $\{0,1,\dots,2^{n-1}-1\}^{(2)}$.

If $z = 2r$ for some $r \in \{0,1,\dots,2^{n-1}-1\}$, then it follows that $y = 2k$ for some $k \in \{0,1,\dots,2^{n-1}-1\}$; consequently, the congruence $f(x,y) \equiv z \pmod{2^n}$ implies the congruence $x + 4k^3 \equiv r \pmod{2^{n-1}}$. Again the function $\psi(x,k)$ in the left hand part of the latter congruence is equiprobable in view of $3.8, 2^\circ$, since $d\psi \equiv dx \pmod 2$ vanishes modulo 2 at no point of $(\mathbb{Z}/2)^{(2)}$ and since the function $\psi \equiv x \pmod 2$ is equiprobable modulo 2. From here, using an argument similar to that of the previous case, we conclude that the congruence $f(x,y) \equiv 2r \pmod{2^n}$ in unknowns x, y has exactly 2^n solutions in $\{0,1,\dots,2^n-1\}^{(2)}$. Thus, f is equiprobable.

Now we are to going to study asymptotically ergodic functions in the class of all uniformly differentiable modulo p functions that have integer-valued derivatives modulo p. It turnes out that these functions could be univariate only. To be more exact, the following theorem is true.

3.11 Theorem. *Let the function $F = (f_1, \dots, f_n) \colon \mathbb{Z}_p^{(n)} \to \mathbb{Z}_p^{(n)}$ be asymptotically ergodic and uniformly differentiable modulo p, and let it have integer-valued derivatives modulo p. Then $n = 1$.*

We will need two lemmata.

3.12 Lemma. *Let the function $f \colon \mathbb{Z}_p^{(n)} \to \mathbb{Z}_p$ be uniformly differentiable modulo p, let it have integer-valued derivatives modulo p, and let f vanish modulo p^k* (i.e., let it be 0 modulo p^k) *for some $k > N_1(f)$ at all points of $\mathbb{Z}_p^{(n)}$. Then each partial derivative modulo p of the function f vanishes modulo p at all points of $\mathbb{Z}_p^{(n)}$.*

Proof of lemma 3.12. Each function $g_i(x_0, x_1, \dots, x_n) = x_i + x_0 f(x_1, \dots, x_n)$ for arbitrary values of $x_0, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ is a bijective modulo p^k function with respect to the variable $x_i, (i = 1, 2, \dots, n)$. As $k > N_1(g_i) = N_1(f)$, then according to 3.9, g_i asymptotically preserves measure, and thus its derivative modulo p vanishes at no point of $\mathbb{Z}_p$. Moreover, the following is true:

$$\frac{\partial_1}{\partial_1 x_i} g_i(u_0, \dots, u_n) = 1 + u_0 \cdot \frac{\partial_1}{\partial_1 x_i} f(u_1, \dots, u_n) \not\equiv 0 \pmod p \qquad (1)$$

for all $u_0, \dots, u_n \in \mathbb{Z}_p$. If

$$\frac{\partial_1}{\partial_1 x_i} f(u_1, \dots, u_n) \equiv d \not\equiv 0 \pmod{p}$$

for some $u_1, \dots, u_n \in \mathbb{Z}_p$, then choosing u_0 such that $u_0 d \equiv -1 \pmod{p}$ we get a contradiction with (1). This proves the lemma. $\square$

3.13 Lemma. *Let the function $H\colon \mathbb{Z}_p^{(n)} \to \mathbb{Z}_p^{(n)}$ be uniformly differentiable modulo p, and let it have integer-valued derivatives modulo p. If H is bijective modulo p^k and if H induces a trivial permutation modulo p^{k-1}* (i.e., an identity transformation of $(\mathbb{Z}/p^{k-1})^{(n)}$) *for some $k > N_1(H) + 1$, then H induces modulo p^k* (i.e., on $(\mathbb{Z}/p^k)^{(n)}$) *either a trivial permutation, or a permutation of order p.*

Proof of the lemma 3.13. Let G be an arbitrary function which satisfies assumptions of the lemma, and let $N_1(G) = N_1(H)$. Represent both H and G in the following form:

$$H(x_1, \dots, x_n) = (x_1, \dots, x_n) + U(x_1, \dots, x_n);$$
$$G(x_1, \dots, x_n) = (x_1, \dots, x_n) + V(x_1, \dots, x_n).$$

Then both U and V are uniformly differentiable modulo p, have integer-valued derivatives modulo p, and $N_1(U) = N_1(V) = N_1(H)$. Moreover, both U and V vanish modulo p^{k-1} on $\mathbb{Z}_p^{(n)}$, for $k - 1 > N_1(H)$. Then lemma 3.12 implies that $U_1' = V_1' = 0$ at all points of $\mathbb{Z}_p^{(n)}$. As $\|U\|_p \le p^{-k+1}$ and $\|V\|_p \le p^{-k+1}$ everywhere on $\mathbb{Z}_p^{(n)}$, then applying 2.4, for all $h_1, \dots, h_n \in \mathbb{Z}_p$ we obtain consequently that

$$\begin{aligned} H(G(h_1, \dots, h_n)) &= H((h_1, \dots, h_n) + V(h_1, \dots, h_n)) \\ &\equiv H(h_1, \dots, h_n) + V(h_1, \dots, h_n) H_1'(h_1, \dots, h_n) \\ &\equiv H(h_1, \dots, h_n) + V(h_1, \dots, h_n) + V(h_1, \dots, h_n) U_1'(h_1, \dots, h_n) \\ &\equiv (h_1, \dots, h_n) + U(h_1, \dots, h_n) + V(h_1, \dots, h_n) \pmod{p^k}. \end{aligned}$$

This implies, in particular, that for all $s \in \mathbb{N}$ the following congruence holds:

$$\begin{aligned} H^s(h_1, \dots, h_n) &= \underbrace{H(\dots H}_{s \text{ times}}(h_1, \dots, h_n)\dots) \\ &\equiv (h_1, \dots, h_n) + sU(h_1, \dots, h_n) \pmod{p^k}. \end{aligned}$$

As U vanishes modulo p^{k-1} everywhere, the latter congruence implies that $H^p(h_1, \dots, h_n) \equiv (h_1, \dots, h_n) \pmod{p^k}$ for all $h_1, \dots, h_n \in \mathbb{Z}_p$. This proves the lemma. $\square$

Proof of theorem 3.11. Choose $k > N_1(F) + 1$ such that F is transitive modulo p^n for all $n \ge k - 1$. The function F induces a permutation on $(\mathbb{Z}/p^k)^{(n)}$, which we denote as $\sigma_k(F)$. Consider a permutation $\sigma = \sigma_k(F)^{p^{(k-1)n}}$. As F is transitive modulo p^k, the order of σ is p^n (and hence σ is not trivial).

On the other hand, $\sigma = \sigma_k(F^{p^{(k-1)n}})$. But $F^{p^{(k-1)n}}$ is bijective modulo p^k and induces a trivial permutation modulo p^{k-1} (the latter assertion follows from the

transitivity of F modulo p^{k-1}). Since σ is not trivial, in view of 3.13 the order of σ must be p. Yet according to the preceding argument the order of σ is p^n, so necessarily $n = 1$. $\square$

It is still an open problem to characterize asymptotically ergodic functions in the class of all uniformly differentiable modulo p functions that have integer-valued derivatives modulo p, but if we additionally assume that the function is uniformly differentiable modulo p^2 and has integer-valued derivative modulo p^2, a description could be obtained. The method we prove the following theorem is in fact a generalization to a p-adic case of the idea originally applied by M. V. Larin in his description of transitive modulo n polynomials over $\mathbb{Z}$, [15].

3.14 Theorem. *Let the function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ be uniformly differentiable modulo p^2 and let f have integer-valued derivative modulo p^2. Then f is asymptotically ergodic if and only if it is transitive modulo $p^{N_2(f)+1}$ for odd prime p or, respectively, modulo $2^{N_2(f)+2}$ for $p = 2$.*

We need the following

3.15 Lemma. *Let the function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ be uniformly differentiable modulo p, and let it have integer-valued derivative modulo p. If f is transitive modulo p^k for some $k > N_1(f)$, then f induces on $\mathbb{Z}/p^{k+1}$ a permutation that is either a single cycle of length p^{k+1} or a product of p pairwise disjoint cycles of length p^k each.*

Proof of lemma 3.15. For $i = 0, 1, 2, \dots$ we denote $x_i = \delta_i(x) \in \{0, 1, \dots, p-1\}$, the ith digit in a canonical representation of a p-adic integer $x \in \mathbb{Z}_p$. Now the definition of uniform differentiability modulo p implies that for an arbitrary $x \in \mathbb{Z}_p$ and $s \geq N_1(f) = N$ there holds a congruence $f(x_0 + x_1 p + \dots + x_{s-1}p^{s-1} + x_s p^s) \equiv f(x_0 + x_1 p + \dots + x_{s-1}p^{s-1}) + x_s p^s f_1'(x_0 + x_1 p + \dots + x_{s-1}p^{s-1}) \pmod{p^{s+1}}$. The latter implies that

$$\delta_s(f(x)) \equiv \Phi_s(x_0, \dots, x_{s-1}) + x_s f_1'(x) \pmod{p}, \tag{1}$$

where $x_i = \delta_i(x) \in \{0, 1, \dots, p-1\}$ is the i-th p-adic digit of $x \in \mathbb{Z}_p, (i = 0, 1, 2, \dots); \Phi_s(x_0, \dots, x_{s-1}) = \delta_s(f(x_0 + x_1 p + \dots + x_{s-1}p^{s-1}))$.

Since a partial derivative $f_1'(x)$ modulo p is periodic with period p^N, $f_1'(x)$ depends only on $x_0, \dots, x_{N-1}$, congruence (1) can be rewritten in the form

$$\delta_s(f(x)) \equiv \Phi_s(x_0, \dots, x_{s-1}) + x_s \Psi(x_0, \dots, x_{N-1}) \pmod{p}, \tag{2}$$

where $\Psi(x_0, \dots, x_{N-1}) = f_1'(x)$. Applying to the composition of functions the rules of derivation modulo p^k, which were mentioned at the beginning of the section, we conclude that for all $r = 1, 2, \dots$ the following congruence holds:

$$(f^r(x))_1' \equiv \prod_{j=0}^{r-1} f_1'(f^j(x)) \pmod{p}. \tag{3}$$

We recall that $f^r(x) = \underbrace{f(\dots f}_{r \text{ times}}(x)\dots), f^0(x) = x$. As f is asymptotically compatible, then transitivity of f modulo p^k for some $k \geq N$ implies transitivity of f

modulo p^n for all $k \geq n \geq N$ (see [11], theorems 2.10 and 1.4). Yet f_1' depends only on $x_0, \dots, x_{N-1}$, and f is transitive modulo p^N, so (3) implies that

$$(f^{p^n}(x))_1' \equiv \left(\prod_{u_0,\dots,u_{N-1}=0}^{p-1} \Psi(u_0, \dots, u_{N-1}) \right)^{p^{n-N}} \pmod{p}. \tag{4}$$

We denote a product in brackets in the right hand part of (4) as Π. Now, since $f^{p^n}(x)$ is uniformly differentiable modulo p and has integer-valued derivative modulo p, in view of (2) and (4) we conclude that

$$\delta_n(f^{p^n}(x)) \equiv \phi_n(x_0, \dots, x_{n-1}) + x_n \Pi^{p^{n-N}} \pmod{p}, \tag{5}$$

where $\phi_n(x_0, \dots, x_{n-1}) = \delta_n(f^{p^n}(x_0 + x_1 p + \dots + x_{n-1}p^{n-1}))$. Since f is a transitive modulo p^{n+1} function for $k \geq n \geq N$, the function f^{p^n}, on the one hand, induces a trivial permutation modulo p^n, and on the other hand, induces on each coset $a + p^n(\mathbb{Z}/p^{n+1})$ of the ring $\mathbb{Z}/p^{n+1}$ a permutation that is a cycle of length p. This in particular means that the function in the right hand part of (5), being considered as a function in a variable x_n, must be a permutation, moreover – a cycle of length p on $\{0, 1, \dots, p-1\}$. It is well known, however, that a polynomial $c + dy \in \mathbb{Z}[y]$ is transitive modulo p iff $d \equiv 1 \pmod{p}$ and $c \not\equiv 0 \pmod{p}$ (see e.g. [2, Ch. 3, Theorem A]). This implies in particular that $\Pi^{p^{n-N}} \equiv 1 \pmod{p}$, and hence $\Pi \equiv 1 \pmod{p}$. Finally we obtain that

$$\begin{aligned} f^{p^k}(x) &\equiv f^{p^k}(x_0 + x_1 p + \dots + x_k p^k) \\ &\equiv x_0 + x_1 p + \dots + x_{k-1}p^{k-1} + p^k(\phi_k(x_0, \dots, x_{k-1}) + x_k) \pmod{p^{k+1}}. \end{aligned} \tag{6}$$

The latter congruence implies that f induces a permutation σ modulo p^{k+1}. We claim that if

$$\phi_k(x_0, \dots, x_{k-1}) \not\equiv 0 \pmod{p}$$

for some (equivalently, all) $x_0, \dots, x_{k-1} \in \{0, 1, \dots, p-1\}$, then f is transitive modulo p^{k+1}; otherwise the permutation σ is a product of exactly p disjoint cycles of length p^k each.

To prove this claim consider some $u_0, \dots, u_k \in \{0, 1, \dots, p-1\}$ and denote C the cycle of the permutation σ that contains the point $u_0 + u_1 p + \dots + u_{k-1}p^{k-1} + x_k p^k \in \mathbb{Z}/p^{k+1}$. Yet f is transitive modulo p^k, so (see (6)) p^k is a factor of $|C|$, the length of the cycle C. If $\phi_k(u_0, \dots, u_{k-1}) \not\equiv 0 \pmod{p}$, then (6) implies that

$$\begin{aligned} f^{p^k}(u_0 + u_1 p + \dots + u_{k-1}p^{k-1} + x_k p^k) \\ \not\equiv u_0 + u_1 p + \dots + u_{k-1}p^{k-1} + x_k p^k \pmod{p^{k+1}}, \end{aligned} \tag{7}$$

i.e., that $|C| > p^k$. On the other hand, (6) implies that $|C|$ is a factor of p^{k+1}. Finally we conclude that in this case $|C| = p^{k+1}$, i.e., f is transitive modulo p^{k+1}.

If $\phi_k(u_0, \dots, u_{k-1}) \equiv 0 \pmod{p}$ holds for some $u_0, \dots, u_k \in \{0, 1, \dots, p-1\}$, then this congruence holds for all $u_0, \dots, u_k \in \{0, 1, \dots, p-1\}$ (otherwise in view of the previous case f is transitive modulo p^{k+1} and (7) holds for all $u_0, \dots, u_k \in$

$\{0, 1, \dots, p-1\}$ and the latter in view of (6) means that $\phi_k(u_0, \dots, u_{k-1}) \not\equiv 0 \pmod p$, a contradiction). Then (6) implies that σ^{p^k} is an identity permutation, i.e. $|C| = p^k$, as p^k is a factor of $|C|$. This proves the lemma. $\square$

Proof of the theorem 3.14. During the proof of the preceding lemma we have established that if f is transitive modulo p^k for some $k \geq N_1(f)$, then f is transitive modulo p^n for all $k \geq n \geq N_1(f)$. So the 'only if' part of the theorem is proved, as $N_2(f) + 1 > N_1(f)$.

Now we have to prove that if $n \geq N_2(f) + 1$ (resp., if $n \geq N_2(f) + 2$ for $p = 2$) and if f is transitive modulo p^n, then it is transitive modulo p^{n+1}. In view of lemma 3.15 it is sufficient to prove that for some $x \in \mathbb{Z}_p$ the following condition holds:

$$f^{p^n}(x) \not\equiv x \pmod{p^{n+1}}. \tag{1}$$

As transitivity modulo p^n implies transitivity modulo p^{n-1}, in view of lemma 3.15 we have

$$f^{p^{n-1}}(x) = x + p^{n-1}\xi(x), \tag{2}$$

where $\xi\colon \mathbb{Z}_p \to \mathbb{Z}_p$ and $\xi(x) \not\equiv 0 \pmod p$ for all $x \in \mathbb{Z}_p$ (otherwise 3.15 implies that f is not transitive modulo p^n, a contradiction to the assumption).

Further, since f is uniformly differentiable modulo p^2 and has integer-valued derivative modulo p^2, we conclude that for all $r = 1, 2, \dots$ the composition f^r is uniformly differentiable modulo p^2 and has integer-valued derivative modulo p^2, and $(f^r(x))'_2 \equiv \prod_{j=0}^{r-1} f'_2(f^j(x)) \pmod{p^2}$ (see (3) of 3.15). Now, since $n-1 \geq N_2(f)$, in view of these considerations and an obvious equality (which follows from (2)) $f^{sp^{n-1}}(x) = f^{(s-1)p^{n-1}}(x + p^{n-1}\xi(x))$, where $s = 1, 2, \dots$, we successively calculate

$$\begin{aligned} f^{p^n}(x) &\equiv f^{(p-1)p^{n-1}}(x) + p^{n-1}\xi(x) \prod_{j=0}^{(p-1)p^{n-1}-1} f'_2(f^j(x)) \\ &\equiv f^{(p-2)p^{n-1}}(x) + p^{n-1}\xi(x)\left(\prod_{j=0}^{(p-2)p^{n-1}-1} f'_2(f^j(x)) + \prod_{j=0}^{(p-1)p^{n-1}-1} f'_2(f^j(x)) \right) \\ &\equiv \dots \equiv x + p^{n-1}\xi(x)\left(1 + \sum_{i=1}^{p-1} \prod_{j=0}^{(p-i)p^{n-1}-1} f'_2(f^j(x))\right) \pmod{p^{n+1}}. \end{aligned} \tag{3}$$

Yet f'_2 is a periodic function with period $p^{N_2(f)}$ and f is transitive modulo p^{n-1}, so we conclude that for arbitrary $i, j \in \mathbb{N}$ the following congruence holds:

$$f'_2(f^j(x)) \equiv f'_2(f^{j+ip^{n-1}}(x)) \pmod{p^2}.$$

In view of the transitivity of f modulo p^{n-1} the latter congruence implies that

$$\prod_{j=0}^{(p-i)p^{n-1}-1} f'_2(f^j(x)) \equiv \alpha(x)^{p-i} \pmod{p^2},$$

where

$$\alpha(x) = \prod_{j=0}^{p^{n-1}-1} f_2'(f^j(x)).$$

In view of (3) we now conclude that

$$f^{p^n}(x) \equiv x + p^{n-1}\xi(x)\left(1 + \sum_{i=1}^{p-1}\alpha(x)^i\right) \pmod{p^{n+1}}. \tag{4}$$

Again, as f_2' modulo p^2 is periodic with period $p^{N_2(f)}$ and f is transitive modulo p^{n-1} for $n-1 \geq N_2(f)$, then $\alpha(x)$ modulo p^2 does not depend on x. Moreover, we claim that $\alpha(x) \equiv 1 \pmod{p}$.

Indeed, during the proof of 3.15 we have already established that if $k \geq N_1(f)$ and if f is a transitive modulo p^k and uniformly differentiable modulo p function with integer-valued derivative modulo p, then

$$\prod_{j=0}^{p^{N_1(f)}-1} f_1'(f^j(x)) \equiv 1 \pmod{p} \tag{5}$$

for all $x \in \mathbb{Z}_p$ (see the proof of (6) in 3.15). The the definition of a derivative modulo p^2 implies that $f_2'(x) \equiv f_1'(x) \pmod{p}$; consequently,

$$\alpha(x) \equiv 1 + p\beta \pmod{p^2} \tag{6}$$

for some $\beta \in \mathbb{N}_0$. In view of (5) and (6), now (4) implies that

$$f^{p^n}(x) \equiv x + p^{n-1}\xi(x)\left(p + p\beta\sum_{i=1}^{p-1} i\right) \pmod{p^{n+1}}, \tag{7}$$

and for $p \neq 2$ we conclude that

$$f^{p^n}(x) \equiv x + p^n\xi(x) \pmod{p^{n+1}}.$$

In view of 3.15 the latter proves the theorem for $p \neq 2$, since $\xi(x) \not\equiv 0 \pmod{p}$ (see (2) and the text thereafter).

When $p = 2$, congruence (7) implies that

$$f^{2^n}(x) \equiv x + 2^n(1+\beta) \pmod{2^{n+1}} \tag{8}$$

and to finish the proof it is sufficient to show that β is even.

For $n \geq N_2(f) + 2$ the transitivity of f modulo 2^n implies that f is transitive modulo $2^{N_2(f)+2}$, so in view of the definition of a derivative modulo p^2 we have that

$$f^{2^N}(x + 2^N\xi) \equiv f^{2^N}(x) + 2^N\xi\prod_{j=0}^{2^N-1} f_2'(f^j(x)) \pmod{2^{N+2}} \tag{9}$$

for $N = N_2(f)$, $\xi \in \mathbb{Z}_2$. Since f is transitive modulo 2^{N+2}, we conclude that for arbitrary $x \in \{0, 1, \dots, 2^N - 1\}$ and with ξ ranging over $\{0, 1, 2, 3\}$ the mapping

$$\phi_x : \xi \mapsto \delta_N(f^{2^N}(x + 2^N\xi)) + 2\delta_{N+1}(f^{2^N}(x + 2^N\xi))$$

is a cycle of length 4 on $\mathbb{Z}/4$. In view of (6),

$$\prod_{j=0}^{2^N-1} f_2'(f^j(x)) \equiv 1 + 2\beta \pmod 4;$$

so (9) implies that

$$\phi_x(\xi) \equiv c(x) + \xi(1 + 2\beta) \pmod 4, \tag{10}$$

where $c(x) = \delta_N(f^{2^N}(x)) + 2\delta_{N+1}(f^{2^N}(x))$. But for each x the mapping ϕ_x is transitive modulo 4, so (10) in view of the above mentioned transitivity criterion for polynomials of degree 1 (see [2 , Ch. 3, Theorem A]) implies that $\beta \equiv 0 \pmod 2$. $\square$

Note. Theorem 3.14 may not hold for a function that is uniformly differentiable modulo p. Namely, for each $n \in \mathbb{N}$ there exists a function $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ such that f is uniformly differentiable modulo 2 and compatible, $f_1' = 1$ everywhere on $\mathbb{Z}_2$, $N_1(f) = 1$, f is transitive modulo 2^k for $k = 1, 2, \dots, n$, and f is not transitive modulo 2^k for all $k > n$. (By the argument similar to those of the following text one can construct a counterexample for $p \neq 2$ as well.)

Represent $x \in \mathbb{Z}_2$ in its canonical 2-adic form $x = x_0 + x_1 \cdot 2 + x_2 \cdot 2^2 + \dots$, $x_0, x_1, x_2 \dots \in \{0, 1\}$. Consider a function

$$f(x) = \sum_{i=0}^{\infty} \phi_i(x_0, \dots, x_i) \cdot 2^i,$$

where each $\phi_i(x_0, \dots, x_i)$ is a Boolean polynomial that is linear with respect to the variable x_i. In other words, $\phi_i(x_0, \dots, x_i) = \psi_i(x_0, \dots, x_{i-1}) + x_i$ in the ring $\mathbb{B}[x_0, \dots, x_i]$ of all Boolean polynomials in variables $x_0, \dots, x_i$. The latter ring is a factor-ring $\mathbb{Z}/2[x_0, \dots, x_i]/(x_0^2 - x_0, \dots, x_i^2 - x_i)$ of the ring $\mathbb{Z}/2[x_0, \dots, x_i]$ of all polynomials in variables $x_0, \dots, x_i$ over $\mathbb{Z}/2$ with respect to the ideal $(x_0^2 - x_0, \dots, x_i^2 - x_i)$ generated by the polynomials $x_0^2 - x_0, \dots, x_i^2 - x_i$ (we assume $\psi_0 = 1$). It is not difficult to see that this function f is compatible (see 3.9 of [11]). Direct calculations show that for arbitrary $s \in \mathbb{N}$ and $h \in \mathbb{Z}_2$ there holds a congruence $f(x + 2^s h) \equiv f(x) + 2^s h \pmod{2^{s+1}}$, i.e., that the function f is uniformly differentiable modulo 2, and $f_1' = 1$ everywhere on $\mathbb{Z}_2$, $N_1(f) = 1$.

Further, in the theory of Boolean functions there are well known sufficient and necessary conditions for transitivity modulo 2^n of the function f of the considerd kind: Namely, f is transitive modulo 2^n iff $\phi_i(x_0, \dots, x_i) = \psi_i(x_0, \dots, x_{i-1}) + x_i$ for $i = 1, 2, \dots, n-1$, where $\psi_0 = 1$, and each Boolean polynomial $\psi_i(x_0, \dots, x_{i-1})$ for $i = 1, 2, \dots, n-1$ is of odd weight (that is, the number of all Boolean vectors $(y_0, \dots, y_{i-1}) \in (\mathbb{Z}/2)^{(n)}$ such that $\psi_i(y_0, \dots, y_{i-1}) \equiv 1 \pmod 2$, is odd). The latter result, which is known as a transitivity modulo 2^n criterion for triangle

transformations, is the mathematical folklore, so it is difficult to attribute it, yet a proof can be found in, e.g., [11], see 4.8 there).

Now taking for the given $n \in \mathbb{N}$ a function f such that $\psi_0 = 1$, the corresponding Boolean polynomials $\psi_i(x_0, \dots, x_{i-1})$ are of odd weight for $i = 1, 2, \dots, n-1$, and the Boolean polynomial $\psi_n(x_0, \dots, x_{n-1})$ is of even weight, we obtain a function that is transitive modulo 2^k for $k = 1, 2, \dots, n$, but is not transitive modulo 2^{n+1}. Thus f is not transitive modulo 2^k for all $k > n$, since (in view of the compatibility of f) the transitivity of f modulo some 2^{k+1} implies its transitivity modulo 2^k.

3.16 Corollary. *Let $A = \langle \mathbb{Z}_p; \Omega \rangle$ be a universal algebra of finite signature Ω, and let all operations of Ω be uniformly differentiable modulo p^2 functions with integer-valued derivatives modulo p^2. Then there exists a positive rational integer $k(A)$ such that every polynomial $f(x) \in A[x]$ is asymptotically ergodic if and only if it is transitive modulo $p^{k(A)}$.*

Proof. The proof is similar to those of 3.10, 3° and thus is omitted. We can take $k(A) = \max\{N_2(\omega) : \omega \in \Omega\} + \epsilon$, where $\epsilon = 1$ if p is odd, otherwise $\epsilon = 2$. □

4. Where Hensel lifting starts.

The results of the preceding section show that for the class $\mathcal{D}_1$ (respectively, $\mathcal{D}_2$) of all uniformly differentiable modulo p (respectively, modulo p^2) functions that have integer-valued derivatives modulo p (respectively, modulo p^2), there exists a function $\zeta\colon \mathcal{D}_1 \to \mathbb{N}$ (respectively, $\eta\colon \mathcal{D}_2 \to \mathbb{N}$) such that the function $f \in \mathcal{D}_1$ (respectively, $f \in \mathcal{D}_2$) is asymptotically measure-preserving (or is ergodic) iff it is bijective (respectively, transitive) modulo $p^{\zeta(f)}$ (respectively, modulo $p^{\eta(f)}$). Theorems 3.9 and 3.14 give corresponding bounds for $\zeta(f)$ and $\eta(f)$.

These bounds are sharp, i.e., there exist a compatible function $f \in \mathcal{D}_1$ (respectively, $f \in \mathcal{D}_2$) such that f is bijective (respectively, transitive) modulo $p^{N_1(f)}$ (respectively, modulo $p^{N_2(f)}$ for $p \neq 2$, or modulo $2^{N_2(f)+1}$ for $p = 2$), but f is not measure-preserving (respectively, is not ergodic). For instance, a polynomial $f(x) = 1 + x^p$ is bijective modulo p, $N_1(f) = 1$, but in force of 3.10, 1° the polynomial f is not bijective modulo p^2 since $f'(z) \equiv 0 \pmod{p}$ for all $z \in \mathbb{Z}_p$.

A corresponding example for theorem 3.14 in case $p \neq 2$ gives a function $f(x) = (x+1) \odot_p 1$, where $\odot_p$ is digitwise multiplication modulo p of p-adic integers: $\delta_i(x \odot_p y) \equiv \delta_i(x)\delta_i(y) \pmod{p}$ for all $i \in \mathbb{N}_0$. The function f is uniformly differentiable, its derivative is 0 everywhere on $\mathbb{Z}_p$, and $N_2(f) = 1$; at the same time f is transitive modulo p, but it is not even bijective (hence, is not transitive) modulo p^2.

Nevertheless, the bounds for $\zeta(f)$ and $\eta(f)$, of, respectively, theorems 3.9 and 3.14, might differ significantly from the ones for functions of various proper subclasses of $\mathcal{D}_1$ and of $\mathcal{D}_2$. For instance, for the function $f(x) = (ax + b) \operatorname{XOR} c$, with $a, b, c \in \mathbb{N}$, theorem 3.14 says that $f(x)$ is asymptotically ergodic iff it is transitive modulo $2^{\lfloor \log_2 c \rfloor + 2}$ since f is uniformly differentiable, its derivative is a everywhere on $\mathbb{Z}_2$, and $N_2(f) = \lfloor \log_2 c \rfloor$. Yet direct application of the above mentioned criterion of transitivity modulo 2^n for triangle transformations together with the already mentioned transitivity criterion for polynomials of degree 1 over $\mathbb{Z}$ immediately implies that f is ergodic iff it is transitive modulo 4. So the problem of sharpening estimates of $\zeta(f)$ and $\eta(f)$ for various important classes, which are smaller then $\mathcal{D}_1$ and $\mathcal{D}_2$, could be of interest.

In this section we study a class $\mathcal{A}$ of all compatible functions $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ such that, loosely speaking, coefficients of their interpolation series tends to 0 as fast as $i!$, or faster (recall that $\lim\limits_{i\to\infty}^{p} i! = 0$). More accurate, a function f, represented by interpolation series ($\diamondsuit$) (see section 2) with p-adic integer coefficients a_i, belongs to $\mathcal{A}$ iff it is compatible, and the sequence $\{\|\frac{a_i}{i!}\|_p : i = 0, 1, 2, \dots\}$ is bounded, i.e., $\|\frac{a_i}{i!}\|_p \leqslant p^{\rho(f)}$ for some $\rho(f) \in \mathbb{N}_0$. Recall that according to theorem 2.1 a function f represented by ($\diamondsuit$) is compatible iff $\|a_i\|_p \leq p^{-\lfloor \log_p i \rfloor}$ for all $i \in \mathbb{N}$.

The class $\mathcal{A}$ is rather wide: It contains all integer-valued compatible analytic on $\mathbb{Z}_p$ functions, in particular, those compatible functions that could be defined by integer-valued polynomials over $\mathbb{Q}_p$. It is known (see [3, Ch. 4, Theorem 4, p. 224]), that a function f of the form ($\diamondsuit$) is analytic on $\mathbb{Z}_p$ iff $\lim\limits_{i\to\infty}^{p} \frac{a_i}{i!} = 0$.

So for the rest of this section we assume that $f \in \mathcal{A}$. Put

$$\lambda(f) = \min\left\{k \in \mathbb{N} : 2\frac{p^k - 1}{p - 1} - k > \rho(f)\right\}.$$

The following theorem is true.

4.1 Theorem. *Let $f \in \mathcal{A}$ and p is an odd prime. The function f is ergodic if and only if it is transitive modulo $p^{\lambda(f)+1}$ (if $p \neq 3$) or modulo $3^{\lambda(f)+2}$ (if $p = 3$).*

Since f is compatible, in view of 2.1 one could represent it in the following form:

$$f(x) = b_0 + \sum_{i=1}^{\infty} b_i p^{\lfloor \log_p i \rfloor} \binom{x}{i},$$

where $b_j \in \mathbb{Z}_p$ for $j = 0, 1, 2, \dots$. Everywhere during the proof we assume that f is represented in this form. Further $\lambda(f)$ is denoted as λ and p is assumed to be an odd prime. We will need some additional technical results.

4.2 Lemma. *Under the assumptions of theorem 4.1 the following is true:*

$$b_i \equiv 0 \pmod{p}, \text{ for } i \geq 2p^{\lambda};$$
$$b_i \equiv 0 \pmod{p^2}, \text{ for } i \geq 3p^{\lambda}.$$

Proof of lemma 4.2. If $b_i = 0$, then the assertion of the lemma is trivial. Suppose that $b_i \neq 0$. Represent f as

$$f(x) = b_0 + \sum_{i=1}^{\infty} \frac{1}{i!} b_i p^{\lfloor \log_p i \rfloor} (x)_i,$$

where, we recall, $(x)_i = x(x-1)\cdots(x-i+1)$ (with $(x)_0 = 1$) is the ith descending factorial power of x. As $f \in \mathcal{A}$, i.e., as

$$\left\| b_i p^{\lfloor \log_p i \rfloor} \right\|_p \leq p^{\rho(f)} \|i!\|_p$$

we conclude that

$$\operatorname{ord}_p b_i \geq \operatorname{ord}_p i! - \left\lfloor \log_p i \right\rfloor - \rho(f), \tag{1}$$

for all $i = 1, 2, \ldots$. We recall that $\log_p \|a\|_p = -\operatorname{ord}_p a$, for $a \in \mathbb{Z}_p$. Thus, the maximal p-prime factor of a is exactly $p^{\operatorname{ord}_p a}$.

We claim that the function $\kappa(i) = \operatorname{ord}_p i! - \left\lfloor \log_p i \right\rfloor$ does not decrease. To prove this we first note that, obviously, $\operatorname{ord}_p i! \geq \operatorname{ord}_p (i-1)!$; so if $\left\lfloor \log_p i \right\rfloor = \left\lfloor \log_p (i-1) \right\rfloor$ then $\kappa(i-1) \leq \kappa(i)$. Assume now that $\left\lfloor \log_p j \right\rfloor > \left\lfloor \log_p (j-1) \right\rfloor$ for some positive rational integer j. Evidently, $\left\lfloor \log_p j \right\rfloor + 1$ is the number of significant digits in the base-p expansion of j. Hence our assumption holds if and only if $j - 1 = (p-1) + (p-1)p + \cdots + (p-1)p^n = p^{n+1} - 1$ for some $n \in \mathbb{N}_0$. But then $\operatorname{ord}_p j! = \operatorname{ord}_p (j-1)! + n$, $\left\lfloor \log_p (j-1) \right\rfloor = n$, $\left\lfloor \log_p j \right\rfloor = n+1$, and so $\kappa(j) > \kappa(j-1)$.

Now to finis the proof of the lemma it is sufficient to show only that $\kappa(2p^\lambda) - \rho(f) \geq 1$ and $\kappa(3p^\lambda) - \rho(f) \geq 2$. We recall that $\operatorname{ord}_p i! = \frac{1}{p-1}(i - \operatorname{wt}_p i)$, where $\operatorname{wt}_p i$ is the sum of all digits in a base-p expansion of i (i.e., by definition, $\operatorname{wt}_p i = i_0 + \cdots + i_s$, where $i = i_0 + i_1 p + \cdots + i_s p^s$, $i_0, \ldots, i_s \in \{0, 1, \ldots, p-1\}$), see e.g., [6 , ch.1, section 2, exercise 13].

As $p \neq 2$, we conclude that $\kappa(2p^\lambda) - \rho(f) = \frac{1}{p-1}(2p^\lambda - 2) - \lambda - \rho(f) \geq 1$ in view of the definition of $\lambda = \lambda(f)$. Hence, if $p \neq 3$, then

$$\kappa(3p^\lambda) - \rho(f) = \frac{1}{p-1}(3p^\lambda - 3) - \lambda - \rho(f) = \kappa(2p^\lambda) + \frac{1}{p-1}(p^\lambda - 1) - \rho(f) \geq 2.$$

This proves the lemma for $p \neq 3$.

Finally, let $p = 3$. Then

$$\kappa(3p^\lambda) - \rho(f) = \kappa(3^{\lambda+1}) - \rho(f) = \frac{1}{2}(3^{\lambda+1} - 1) - \lambda - 1 - \rho(f) \geq 2,$$

otherwise in view of the inequality

$$3^\lambda - 1 - \lambda > \rho(f),$$

(which follows directly from the definition of $\lambda = \lambda(f)$) we get

$$\frac{1}{2}(3^{\lambda+1} - 1) - \lambda - 1 - 3^\lambda + 1 + \lambda < 1,$$

i.e., $3^\lambda - 1 < 2$, and so $\lambda < 1$, a contradiction. This finishes the proof. □

4.3 Corollary. *Under the assumptions of theorem 4.2, for $i \in \mathbb{N}$ the following is true:*

$$\frac{\Delta^i f(x)}{i} \equiv \begin{cases} 0 \pmod{p^2}, & \text{if } i \geq 2p^\lambda + 1; \\ 0 \pmod{p}, & \text{if } i \geq p^\lambda + 1. \end{cases}$$

Proof of corollary 4.3. As $\Delta^j \binom{x}{i} = \binom{x}{i-j}$ if $i \geq j$ and $\Delta^j \binom{x}{i} = 0$ if $i < j$, then

$$\frac{\Delta^i f(x)}{i} = \frac{1}{\hat{\imath}} \sum_{j=i}^{\infty} b_j p^{\lfloor \log_p j \rfloor - \operatorname{ord}_p j} \binom{x}{j-i},$$

where $\hat{\imath} = i p^{-\operatorname{ord}_p i} \in \mathbb{Z}_p, \operatorname{ord}_p \hat{\imath} = 0$. Now the result is obvious in view of lemma 4.2. □

4.4 Proposition. *Under assumptions of theorem 4.1 the function f is uniformly differentiable modulo p^2, has integer-valued derivative modulo p^2, and $N_2(f) \le \lambda(f)+1$. Moreover,*

$$f_2'(x) \equiv \sum_{i=1}^{2p^\lambda} (-1)^{i-1} \frac{\Delta^i f(x)}{i} \pmod{p^2}.$$

Proof of proposition 4.4. To prove the first assertion of the proposition we will demonstrate that there exists a function $f_2'\colon \mathbb{Z}_p \to \mathbb{Z}_p$ such that for all $x, h \in \mathbb{Z}_p$ and $m \ge \lambda(f)+1$ the following congruence holds:

$$f(x+p^m h) \equiv f(x) + p^m h f_2'(x) \pmod{p^{m+2}}. \tag{1}$$

In view of the compatibility of f, it is sufficient to prove the congruence (1) only for $h \in \{1, 2, \dots, p^2-1\}$ (for $h=0$ the congruence is trivial). Applying the Newton's formula

$$f(x+n) = \sum_{i=0}^{n} \binom{n}{i} \Delta^i f(x)$$

for $n = p^m h$, we have

$$f(x+p^m h) = f(x) + p^m h \phi_m(x,h), \tag{2}$$

where

$$\phi_m(x,h) = \sum_{i=1}^{p^m h} \binom{p^m h - 1}{i-1} \frac{\Delta^i f(x)}{i}. \tag{3}$$

Hence in view of 4.3 for $m \ge \lambda + 1$ we obtain:

$$\phi_m(x,h) \equiv \sum_{i=1}^{2p^\lambda} \binom{p^m h - 1}{i-1} \frac{\Delta^i f(x)}{i} \pmod{p^2}. \tag{4}$$

Further, for $i = 1, 2, \dots, 2p^\lambda$ the following obviuos equality hold:

$$\binom{p^m h - 1}{i-1} = \prod_{k=0}^{i-2} \frac{p^m h - (k+1)}{k+1} = \prod_{j=1}^{i-1} \left(\frac{h}{\hat{j}} p^{m - \operatorname{ord}_p j} - 1 \right). \tag{5}$$

Here $\hat{j} = j p^{-\operatorname{ord}_p j}$ is the unit of $\mathbb{Z}_p$, i.e., $\hat{j}$ has a multiplicative inverse $\frac{1}{\hat{j}}$ in $\mathbb{Z}_p$; hence, each factor of the product in the right hand part of (5) is a p-adic integer.

If $i \le p^\lambda$ then $m - \operatorname{ord}_p j \ge 2$ for all $j = 1, 2, \dots, i-1$; so (5) implies that

$$\binom{p^m h - 1}{i-1} \equiv (-1)^{i-1} \pmod{p^2}. \tag{6}$$

If $p^\lambda + 1 \le i \le 2p^\lambda$ and $j \in \{1, 2, \dots, i-1\}$ then $m - \operatorname{ord}_p j = 1$ only in the case when simultaneously $j = p^\lambda$ and $m = \lambda + 1$ hold; otherwise $m - \operatorname{ord}_p j \ge 2$. Yet if $m - \operatorname{ord}_p j = 1$ then

$$\frac{\Delta^i f(x)}{i} \equiv 0 \pmod{p}$$

(see 4.3); hence in both cases we have that

$$\left(\frac{h}{\hat{j}} p^{m-\operatorname{ord}_p j} - 1\right) \frac{\Delta^i f(x)}{i} \equiv -\frac{\Delta^i f(x)}{i} \pmod{p^2}.$$

So in view of (5) we conclude that

$$\binom{p^m h - 1}{i-1} \frac{\Delta^i f(x)}{i} \equiv (-1)^{i-1} \frac{\Delta^i f(x)}{i} \pmod{p^2}. \tag{7}$$

for all $i = 1, 2, \dots, 2p^\lambda$. Now combining together (4), (6), and (7) we conclude that

$$\phi_m(x, h) \equiv \sum_{i=1}^{2p^\lambda} (-1)^{i-1} \frac{\Delta^i f(x)}{i} \pmod{p^2};$$

this in view of (2), (3), and (4) completes the proof of proposition 4.4. □

4.5. Lemma. *Under assumptions of theorem 4.1, there exists a function $\theta\colon \mathbb{Z}_p \to \mathbb{Z}_p$ such that for arbitrary $x, h \in \mathbb{Z}_p$ the following congruence holds:*

$$f(x + p^\lambda h) \equiv f(x) + p^\lambda h f_2'(x) + p^{\lambda+1} h^2 \theta(x) \pmod{p^{\lambda+2}}.$$

The function θ satisfies the following condition: For arbitrary $a, b \in \mathbb{Z}_p$ the congruence $a \equiv b \pmod{p^\lambda}$ implies $\theta(a) \equiv \theta(b) \pmod{p}$. Moreover, one may take

$$\theta(x) = \sum_{j=2}^{p-1} (-1)^j \sum_{i=1}^{j-1} \frac{1}{i} \frac{\Delta^{jp^{\lambda-1}} f(x)}{jp^{\lambda-1}} + \sum_{k=1}^{p-1} (-1)^{k-1} \frac{\Delta^{kp^{\lambda-1}+p^\lambda} f(x)}{kp^\lambda} + \frac{\Delta^{2p^\lambda} f(x)}{2p^{\lambda+1}}.$$

Proof of lemma 4.5. We start proving that the function θ defined by the latter equality is integer-valued on $\mathbb{Z}_p$. Since f is compatible, each fraction $\frac{\Delta^s f(x)}{s}$ for $s = 1, 2, 3, \dots$, is a p-adic integer (see 3.1 of [11]); thus it is sufficient to prove only that for all $k \in \{1, 2, \dots, p-1\}$ all the following functions $\alpha(x)$ and $\beta_k(x)$ are integer-valued on $\mathbb{Z}_p$:

$$\alpha(x) = \frac{\Delta^{2p^\lambda} f(x)}{2p^{\lambda+1}}; \quad \beta_k(x) = \frac{\Delta^{kp^{\lambda-1}+p^\lambda} f(x)}{kp^\lambda}.$$

Since

$$\Delta^i f(x) = \sum_{j=i}^{\infty} b_j p^{\lfloor \log_p j \rfloor} \binom{x}{j-i} \tag{1}$$

for $i = 1, 2, 3, \dots$ and

$$b_j p^{\lfloor \log_p j \rfloor} \equiv 0 \pmod{p^{\lambda+1}}$$

for all integer rationals $j \geq 2p^\lambda$ (see 4.2), then $\alpha(x) \in \mathbb{Z}_p$. If $j \geq kp^{\lambda-1} + p^\lambda$ then $\lfloor \log_p j \rfloor \geq \lambda$; hence (1) implies that $\beta_k(x) \in \mathbb{Z}_p$.

Now we prove that for all $a, b \in \mathbb{Z}_p$ the congruence $a \equiv b \pmod{p^\lambda}$ implies $\theta(a) \equiv \theta(b) \pmod{p}$. In view of (1) and 4.2 the following congruence holds:

$$\alpha(x) \equiv \frac{1}{2} \sum_{j=2p^\lambda}^{3p^\lambda - 1} \frac{1}{p} b_j \binom{x}{j - 2p^\lambda} \pmod{p}. \tag{2}$$

We recall the well-known Lucas' theorem (for a proof see e.g [4]): If $a = \sum_{i=0}^{\infty} a_i p^i$ and $b = \sum_{i=0}^{N} b_i p^i$ are, respectively, canonical representations of a p-adic integer a and of a non-negative integer rational b (i.e., $a_i, b_i \in \{0, 1, \dots, p-1\}$ for $i = 0, 1, 2, \dots$), then

$$\binom{a}{b} \equiv \binom{a_0}{b_0} \binom{a_1}{b_1} \cdots \binom{a_N}{b_N} \pmod{p}.$$

So, if $a \equiv b \pmod{p^\lambda}$, then the Lucas' theorem implies that for all $j = 2p^\lambda, 2p^\lambda + 1, \dots, 3p^\lambda - 1$ the following congruence holds:

$$\binom{a}{j - 2p^\lambda} \equiv \binom{b}{j - 2p^\lambda} \pmod{p}.$$

Thus, (2) implies that

$$\alpha(a) \equiv \alpha(b) \pmod{p}. \tag{3}$$

Further, combining (1) with 4.2 we obtain that

$$\beta_k(x) \equiv \frac{1}{k} \sum_{j=kp^{\lambda-1}+p^\lambda}^{2p^\lambda - 1} b_j \binom{x}{j - kp^{\lambda-1} - p^\lambda} \pmod{p}$$

for all $k = 1, 2, \dots, p-1$. Now applying the Lucas' theorem once again we conclude that

$$\beta_k(a) \equiv \beta_k(b) \pmod{p} \tag{4}$$

for $a \equiv b \pmod{p^\lambda}$.

Further, assuming that

$$\gamma_k(x) = \frac{\Delta^{kp^{\lambda-1}} f(x)}{kp^{\lambda-1}},$$

in view of (1) we conclude that for $k = 1, 2, \dots, p-1$ the following congruence holds:

$$\gamma_k(x) \equiv \frac{1}{k} \sum_{j=kp^{\lambda-1}}^{p^\lambda - 1} b_j \binom{x}{j - kp^{\lambda-1}} \pmod{p}.$$

Applying the Lucas' theorem once again we conclude that

$$\gamma_k(a) \equiv \gamma_k(b) \pmod{p} \tag{5}$$

for $a \equiv b \pmod{p^\lambda}$. Hence in view of (3) – (5) the congruence $a \equiv b \pmod{p^\lambda}$ implies the congruence $\theta(a) \equiv \theta(b) \pmod{p}$.

Now we prove the rest of the lemma. As f is compatible, during the proof we may assume that $h \in \mathbb{N}$ (the case $h = 0$ is trivial). According to 4.4 (see (2)–(5) there) the following is true:

$$f(x + p^\lambda h) \equiv f(x) + p^\lambda h \phi(x, h) \pmod{p^{\lambda+2}}, \tag{6}$$

where

$$\phi(x, h) \equiv \sum_{i=1}^{2p^\lambda} \binom{p^\lambda h - 1}{i - 1} \frac{\Delta^i f(x)}{i} \pmod{p^2}. \tag{7}$$

and, besides,

$$\binom{p^\lambda h - 1}{i - 1} = \prod_{j=1}^{i-1} (\frac{h}{\hat{j}} p^{\lambda - \operatorname{ord}_p j} - 1) \tag{8}$$

for $i = 1, 2, \ldots, 2p^\lambda$. As f is compatible, then, according to 3.40 of [11],

$$\frac{\Delta^i f(x)}{i} \equiv 0 \pmod{p}$$

in all cases with the exception of, possibly, a case when i is of the form $i = tp^s$ for suitable $t \in \{1, 2, \ldots p - 1\}$ and $s \in \mathbb{N}_0$. Thus, if $i \leq p^{\lambda-1}$, as well as if simultaneously $p^{\lambda-1} < i < p^\lambda$ and $p^{\lambda-1}$ is not a factor of i, the equality (8) implies:

$$\binom{p^\lambda h - 1}{i - 1} \frac{\Delta^i f(x)}{i} \equiv (-1)^{i-1} \frac{\Delta^i f(x)}{i} \pmod{p^2}. \tag{9}$$

Let $i = kp^{\lambda-1}$ for $k \in \{2, 3, \ldots, p - 1\}$. Then (8) implies:

$$\binom{p^\lambda h - 1}{i - 1} \equiv (-1)^{kp^{\lambda-1}-1} + (-1)^k ph \sum_{j=1}^{k-1} \frac{1}{j} \pmod{p^2}. \tag{10}$$

Further, if $p^\lambda \leq i \leq 2p^\lambda$ and $\operatorname{ord}_p i \neq \lambda, \lambda - 1$ then (1) (together with the congruence that follow it) imply that

$$\frac{\Delta^i f(x)}{i} \equiv 0 \pmod{p^2}. \tag{11}$$

Now we have to study the only two remaining cases: $i = \nu p^\lambda$ for $\nu \in \{1, 2\}$ and $i = kp^{\lambda-1} + p^\lambda$ for $k \in \{1, 2, \dots, p-1\}$. The latter one in view of 4.3 and (8) implies that

$$\binom{p^\lambda h - 1}{i-1} \frac{\Delta^i f(x)}{i} \equiv (-1)^{i-1} \frac{\Delta^i f(x)}{i} + (-1)^{k-1} h \frac{\Delta^i f(x)}{i} \pmod{p^2}. \tag{12}$$

Further, for $k = 1, 2, \dots, p-1$ the following trivial equality holds in $\mathbb{Q}_p$:

$$\left(1 + \frac{p}{k}\right) \frac{\Delta^{kp^{\lambda-1}+p^\lambda} f(x)}{kp^{\lambda-1} + p^\lambda} = \frac{\Delta^{kp^{\lambda-1}+p^\lambda} f(x)}{kp^{\lambda-1}} \tag{13}$$

From here in view of 4.3 we conclude that

$$\frac{\Delta^{kp^{\lambda-1}+p^\lambda} f(x)}{kp^{\lambda-1} + p^\lambda} \equiv 0 \pmod{p}$$

and since $\frac{p}{k} \in \mathbb{Z}_p$ and $\operatorname{ord}_p \frac{p}{k} = 1$, the equality (13) implies that

$$\frac{\Delta^{kp^{\lambda-1}+p^\lambda} f(x)}{kp^{\lambda-1} + p^\lambda} \equiv \frac{\Delta^{kp^{\lambda-1}+p^\lambda} f(x)}{kp^{\lambda-1}} \pmod{p^2}.$$

Hence, applying (12) for $i = kp^{\lambda-1} + p^\lambda$, we have that

$$\begin{aligned} &\binom{p^\lambda h - 1}{kp^{\lambda-1} + p^\lambda - 1} \frac{\Delta^{kp^{\lambda-1}+p^\lambda} f(x)}{kp^{\lambda-1} + p^\lambda} \\ &\qquad = (-1)^{kp^{\lambda-1}+p^\lambda-1} \frac{\Delta^{kp^{\lambda-1}+p^\lambda} f(x)}{kp^{\lambda-1} + p^\lambda} + (-1)^{k-1} ph\beta_k(x) \pmod{p^2}. \end{aligned} \tag{14}$$

In case $i = p^\lambda$, the equality (8) implies that

$$\binom{p^\lambda h - 1}{p^\lambda - 1} \equiv (-1)^{p^\lambda - 1} - ph \sum_{j=1}^{p-1} \frac{1}{j} \equiv (-1)^{p^\lambda - 1} \pmod{p^2}, \tag{15}$$

since for $p \neq 2$ the following congruences hold in $\mathbb{Q}_p$: $\sum_{j=1}^{p-1} \frac{1}{j} \equiv \sum_{j=1}^{p-1} j \equiv 0 \pmod{p}$.

Finally, for $i = 2p^\lambda$, applying (8) and 4.3, we conclude that

$$\begin{aligned} \binom{p^\lambda h - 1}{2p^\lambda - 1} \frac{\Delta^{2p^\lambda} f(x)}{2p^\lambda} &\equiv (-1)^{2p^\lambda - 1} \frac{\Delta^{2p^\lambda} f(x)}{2p^\lambda} + h \frac{\Delta^{2p^\lambda} f(x)}{2p^\lambda} \\ &\equiv (-1)^{2p^\lambda - 1} \frac{\Delta^{2p^\lambda} f(x)}{2p^\lambda} + hp\alpha(x) \pmod{p^2}, \end{aligned} \tag{16}$$

where $\alpha(x) \in \mathbb{Z}_p$, as it was shown above.

Now combining together (6), (7), (9), (11), (14), (15), (16), and 4.4, we finish the proof of lemma 4.5. $\square$

4.6 Lemma. *Under assumptions of theorem 4.1, for all $x, h \in \mathbb{Z}_p$ the following congruence holds:*

$$f_2'(x + p^\lambda h) \equiv f_2'(x) + 2ph\theta(x) \pmod{p^2}.$$

Here θ is the function defined in 4.5.

Proof of lemma 4.6. In view of 4.4 the following is true:

$$f_2'(x + p^\lambda h) \equiv \sum_{i=1}^{2p^\lambda} (-1)^{i-1} \frac{\Delta^i f(x + p^\lambda h)}{i} \pmod{p^2}. \tag{1}$$

For $i = 1, 2, \dots, 2p^\lambda$ the previous lemma implies that

$$\frac{\Delta^i f(x + p^\lambda h)}{i} \equiv \frac{\Delta^i f(x)}{i} + hp^{\lambda - \operatorname{ord}_p i} \frac{\Delta^i f_2'(x)}{\hat{\imath}} + h^2 p^{\lambda + 1 - \operatorname{ord}_p i} \frac{\Delta^i \theta(x)}{\hat{\imath}} \pmod{p^2}, \tag{2}$$

where $\hat{\imath} = ip^{-\operatorname{ord}_p i}$ is a unit in $\mathbb{Z}_p$, i.e., it has a multiplicative inverse $\frac{1}{\hat{\imath}} \in \mathbb{Z}_p$.

The term of order 2 (with respect to h) in (2) may not vanish modulo p^2 only if $i \in \{p^\lambda, 2p^\lambda\}$. Yet, as $\Delta^j \binom{x}{\nu} = \binom{x}{\nu - j}$ for $\nu \geq j$ and $\Delta^j \binom{x}{\nu} = 0$ for $\nu < j$, for all $j \in \mathbb{N}$ we have

$$\Delta^j f(x) = \sum_{\nu=j}^{\infty} b_\nu p^{\lfloor \log_p \nu \rfloor} \binom{x}{\nu - j}. \tag{3}$$

Consequently, if $j \in \{p^\lambda, 2p^\lambda\}$, then

$$\frac{\Delta^{j + kp^{\lambda-1}} f(x)}{kp^{\lambda-1}} \equiv 0 \pmod{p}. \tag{4}$$

for $k \in \{1, 2, \dots, p-1\}$. Further, for $j \in \{p^\lambda, 2p^\lambda\}$ equality (3) in view of 4.3 implies that

$$\frac{\Delta^{j + kp^{\lambda-1} + p^\lambda} f(x)}{kp^\lambda} \equiv 0 \pmod{p}, \tag{5}$$

$$\frac{\Delta^{j + 2p^\lambda} f(x)}{2p^\lambda} \equiv 0 \pmod{p}. \tag{6}$$

Now, by the definition of θ, combining together (4), (5), (6) we conclude that $\frac{\Delta^i \theta(x)}{\hat{\imath}} \equiv 0 \pmod{p}$ for $i \in \{p^\lambda, 2p^\lambda\}$, and thus

$$h^2 p^{\lambda + 1 - \operatorname{ord}_p i} \frac{\Delta^i \theta(x)}{\hat{\imath}} \equiv 0 \pmod{p^2} \tag{7}$$

for all $i = 1, 2, \dots, 2p^\lambda$.

The term of order 1 in (2) may not vanish modulo p^2 only for $i \in \{1, 2, \dots, 2p^\lambda\}$ such that $\operatorname{ord}_p i \geq \lambda - 1$, i.e., for

$$i \in \{p^\lambda, 2p^\lambda, kp^{\lambda-1}, kp^{\lambda-1} + p^\lambda : k = 1, 2, \dots, p-1\}.$$

Combining together 4.3, 4.4, and 3.4 of [11], which we already have referred to (see the argument that follows (8) in the proof of 4.5), we have

$$f_2'(x) \equiv \frac{\Delta^{p^\lambda} f(x)}{p^\lambda} + \sum_{t=0}^{\lambda-1} \sum_{\tau=1}^{p-1} (-1)^{\tau-1} \frac{\Delta^{\tau p^t} f(x)}{\tau p^t} \pmod{p}, \tag{8}$$

and hence

$$\Delta^i f_2'(x) \equiv \frac{\Delta^{i+p^\lambda} f(x)}{p^\lambda} + \sum_{t=0}^{\lambda-1} \sum_{\tau=1}^{p-1} (-1)^{\tau-1} \frac{\Delta^{i+\tau p^t} f(x)}{\tau p^t} \pmod{p}. \tag{9}$$

The latter congruence for $i \in \{kp^{\lambda-1} + p^\lambda : k = 1, 2, \dots, p-1\}$ in force of (3) and 4.2 implies that $\Delta^i f_2'(x) \equiv 0 \pmod{p}$, and consequently

$$hp \frac{\Delta^{kp^{\lambda-1}+p^\lambda} f_2'(x)}{k+p} \equiv 0 \pmod{p^2} \tag{10}$$

for $k = 1, 2, \dots, p-1$ (since the multiplicative inverse $\frac{1}{k+p}$ of $k+p$ is in $\mathbb{Z}_p$).

If $i \in \{kp^{\lambda-1} : k = 1, 2, \dots, p-1\}$ then in view of 4.2, (3) and (9) we obtain that

$$\Delta^{kp^{\lambda-1}} f_2'(x) \equiv \frac{\Delta^{kp^{\lambda-1}+p^\lambda} f(x)}{p^\lambda} + \sum_{\tau=1}^{p-k-1} (-1)^{\tau-1} \frac{\Delta^{(\tau+k)p^{\lambda-1}} f(x)}{\tau p^{\lambda-1}} \pmod{p}. \tag{11}$$

If $i = 2p^\lambda$ then 4.4 implies that

$$\Delta^{2p^\lambda} f_2'(x) \equiv \sum_{j=1}^{2p^\lambda} (-1)^{j-1} \frac{\Delta^{j+2p^\lambda} f(x)}{j} \pmod{p^2}.$$

This in view of (3) and 4.2 implies that

$$\Delta^{2p^\lambda} f_2'(x) \equiv 0 \pmod{p^2}. \tag{12}$$

Now we consider the case $i = p^\lambda$. Proposition 4.4 implies that

$$\Delta^{p^\lambda} f_2'(x) \equiv \sum_{j=1}^{1+p^\lambda} (-1)^{j-1} \frac{\Delta^{j+p^\lambda} f(x)}{j} \pmod{p^2}, \tag{13}$$

since, combining together (3) and 4.2, for $j = p^\lambda + 1, \dots, 2p^\lambda$ we conclude that

$$\frac{\Delta^{j+p^\lambda} f(x)}{j} \equiv 0 \pmod{p^2}.$$

Moreover, (3) implies that the latter congruence holds also for all $j \le p^\lambda - 1$ such that $j \ne kp^{\lambda-1}$, where $k = 1, 2, \dots, p-1$. Thus, (13) implies that

$$\Delta^{p^\lambda} f_2'(x) \equiv \frac{\Delta^{2p^\lambda} f(x)}{p^\lambda} + \sum_{k=1}^{p-1} (-1)^{k-1} \frac{\Delta^{kp^{\lambda-1}+p^\lambda} f(x)}{kp^{\lambda-1}} \pmod{p^2}. \tag{14}$$

Now, substituting (7), (10), (11), (12), (14) to (2) and summarizing up all the obtained congruences for i ranging from 1 to $2p^\lambda$, in view of (1) and 4.4 we conclude that

$$\begin{aligned} f_2'(x + p^\lambda h) \equiv f_2'(x) + hp \Bigg(& \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \sum_{\tau=1}^{p-k-1} (-1)^{\tau-1} \frac{\Delta^{(\tau+k)p^{\lambda-1}} f(x)}{\tau p^{\lambda-1}} \\ & + \sum_{k=1}^{p-1} (-1)^{k-1} \frac{\Delta^{kp^{\lambda-1}+p^\lambda} f(x)}{kp^\lambda} \Bigg) \\ & + h \sum_{k=1}^{p-1} (-1)^{k-1} \frac{\Delta^{kp^{\lambda-1}+p^\lambda} f(x)}{kp^{\lambda-1}} + h \frac{\Delta^{2p^\lambda} f(x)}{p^\lambda} (\mathrm{mod} p^2). \end{aligned} \tag{15}$$

We recall that here and after all calculations are performed in the field $\mathbb{Q}_p$, and by the above agreement the congruence $\xi \equiv 0 \pmod{p^k}$ for $\xi \in \mathbb{Q}_p$ and a positive integer rational k means that $\|\xi\|_p = p^{-k}$ (hence, ξ is a p-adic integer). Proceeding with this note we conclude that for $k, \tau \in \{1, 2, \dots, p-1\}$ the following equalities hold in $\mathbb{Q}_p$:

$$\begin{aligned} & \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \sum_{\tau=1}^{p-k-1} (-1)^{\tau-1} \frac{\Delta^{(\tau+k)p^{\lambda-1}} f(x)}{\tau p^{\lambda-1}} \\ & = \sum_{m=1}^{p-1} (-1)^m \sum_{k+\tau=m} \frac{1}{k\tau} \cdot \frac{\Delta^{mp^{\lambda-1}} f(x)}{p^{\lambda-1}} = 2 \sum_{m=1}^{p-1} (-1)^m \sum_{\tau=1}^{m-1} \frac{1}{\tau} \cdot \frac{\Delta^{mp^{\lambda-1}} f(x)}{mp^{\lambda-1}}, \end{aligned} \tag{16}$$

since for $k, \tau \in \{1, 2, \dots, p-1\}$ it is obvious that

$$\sum_{k+\tau=m} \frac{1}{k\tau} = \sum_{k+\tau=m} \frac{1}{(m-\tau)\tau} = \frac{1}{m} \sum_{k+\tau=m} \left(\frac{1}{\tau} + \frac{1}{m-\tau}\right) = \frac{2}{m} \sum_{\tau=1}^{m-1} \frac{1}{\tau}.$$

Besides, as it was shown during the proof of 4.5, both $\alpha(x)$ and $\beta_k(x)$ are p-adic integers for $k = 1, 2, \dots, p-1$ and $x \in \mathbb{Z}_p$; thus

$$2hp\alpha(x) = h \frac{\Delta^{2p^\lambda} f(x)}{p^\lambda}; \quad hp\beta_k(x) = h \frac{\Delta^{kp^{\lambda-1}+p^\lambda} f(x)}{kp^{\lambda-1}}, \tag{17}$$

where all the factors are p-adic integers. Now the assertion of the lemma follows from (15), (16), (17), and the definition of the function θ. $\square$

Proof of theorem 4.1. To prove theorem 4.1 we first note that according to 4.4 there holds an inequality $N_2(f) \le \lambda(f) + 1$. Thus, by 3.14 it is sufficient only to show

that if $p \neq 3$ and f is transitive modulo $p^{\lambda(f)+1}$ then it is transitive modulo $p^{\lambda(f)+2}$. In turn, for this purpose in view of 3.15 it is sufficient only to prove that

$$f^{p^{\lambda+1}}(x) \not\equiv x \pmod{p^{\lambda+2}} \tag{1}$$

at least for one $x \in \mathbb{Z}_p$. Further we merely calculate $f^{p^{\lambda+1}}(x) \bmod p^{\lambda+2}$.

Under the assumptions we have made above, f is transitive modulo p^λ, since f is compatible. Then by 3.15 we conclude that for all $x \in \mathbb{Z}_p$

$$f^{p^\lambda}(x) = x + p^\lambda \xi(x), \quad \xi(x) \not\equiv 0 \pmod{p}, \tag{2}$$

where $\xi\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is a function defined everywhere on $\mathbb{Z}_p$.

We assert that for each $i = 0, 1, 2, \ldots$ the following congruence holds:

$$\begin{aligned} f^{p^\lambda+i}(x) \equiv f^i(x) + p^\lambda \xi(x) \prod_{j=0}^{i-1} f_2'(f^j(x)) \\ + p^{\lambda+1}\xi(x)^2 \prod_{j=0}^{i-1} f_2'(f^j(x)) \sum_{k=0}^{i-1} \frac{\theta(f^k(x))}{f_2'(f^k(x))} \prod_{\tau=0}^{k-1} f_2'(f^\tau(x)) \pmod{p^{\lambda+2}} \end{aligned} \tag{3}$$

Recall that the sum (resp., product) over the empty set of indexes is assumed to be 0 (resp., 1). Note also that since f is transitive modulo $p^{\lambda+1}$ it is bijective modulo $p^{\lambda+1}$. Consequently, f is bijective modulo $p^\lambda, \ldots, p^2, p$ since f is compatible. Hence $f_1'(x) \not\equiv 0 \pmod{p}$ for all $x \in \mathbb{Z}_p$ (see the proof of 3.9) and thus $f_2'(x) \not\equiv 0 \pmod{p}$ either (as $f_2'(x) \equiv f_1'(x) \pmod{p}$). So all the denominators in (3) have multiplicative inverses in $\mathbb{Z}_p$; thus, during the proof of (3) and further we assume that all the calculations are performed in $\mathbb{Z}_p$.

One could easily prove (3) by the induction on i. If $i = 0$, then (3) trivially follows from (2). Assume that (3) is true for $i = m - 1$. As

$$f^{p^\lambda+m}(x) = f(f^{p^\lambda+m-1}(x)) \tag{4}$$

then, assuming in (3) that $i = m - 1$, substituting (3) to (4), applying 4.5 and a congruence $(f^k(x))_2' \equiv \prod_{j=0}^{k-1} f_2'(f^j(x)) \pmod{p^2}$, we prove congruence (3) for $i = m$, in view of compatibility of f, by obvious direct calculations. We omit details.

Now we apply (3) to calculate $f^{p^{\lambda+1}}(x) \bmod p^{\lambda+2}$. Put

$$A_i(x) = (f^i(x))_2' = \prod_{j=0}^{i-1} f_2'(f^j(x));$$

$$\begin{aligned} B_i(x) = (f^i(x))_2' \sum_{k=0}^{i-1} \frac{(f^k(x))_2'}{f_2'(f^k(x))} \theta(f^k(x)) = \\ = \left(\prod_{j=0}^{i-1} f_2'(f^j(x))\right) \cdot \left(\sum_{k=0}^{i-1} \frac{\theta(f^k(x))}{f_2'(f^k(x))^2} \prod_{\tau=0}^{k} f_2'(f^\tau(x))\right). \end{aligned}$$

Lemma 4.6 implies that

$$f_2'(a+p^\lambda h) \equiv \begin{cases} f_2'(a) \pmod{p^2}, & \text{if } h=0; \\ f_2'(a) \pmod{p}, & \text{if } h\neq 0. \end{cases} \tag{5}$$

As f is transitive modulo p^λ, then (5) implies that $f_2'(f^k(x)) \equiv f_2'(f^r(x)) \pmod{p}$ as soon as $k \equiv r \pmod{p^\lambda}$. Besides, by 4.5 the latter condition implies that $\theta(f^k(x)) \equiv \theta(f^r(x)) \pmod{p}$.

Further,

$$\prod_{\tau=0}^{p^\lambda-1} f_2'(f^\tau(x)) \equiv 1 \pmod{p}. \tag{6}$$

This has been already proven in 3.15 (see the proof of (6) there), since 4.5 implies that $N_1(f) \leq \lambda$. Consequently,

$$\prod_{\tau=0}^{k} f_2'(f^\tau(x)) \equiv \prod_{\tau=0}^{r} f_2'(f^\tau(x)) \pmod{p}$$

as soon as $k \equiv r \pmod{p^\lambda}$.

Finally we conclude that for every $t \in \mathbb{N}$

$$B_{tp^\lambda}(x) \equiv t \sum_{\tau=0}^{p^\lambda-1} \frac{\theta(f^\tau(x))}{f_2'(f^\tau(x))^2} \prod_{\nu=0}^{\tau} f_2'(f^\nu(x)) \equiv tB_{p^\lambda}(x) \pmod{p}. \tag{7}$$

Now we calculate $A_{tp^\lambda}(x) \pmod{p^2}$ for $t \in \mathbb{N}$. Congruence (3) in view of (6) implies that

$$f^{kp^\lambda+\tau}(x) \equiv f^\tau(x) + kp^\lambda \xi(x) \prod_{j=0}^{\tau-1} f_2'(f^j(x)) \pmod{p^{\lambda+1}} \tag{8}$$

for all $k \in \mathbb{N}$ and all $\tau \in \{0,1,\dots,p^\lambda-1\}$. As

$$A_{tp^\lambda}(x) = \prod_{k=0}^{t-1} \prod_{\tau=0}^{p^\lambda-1} f_2'(f^{kp^\lambda+\tau}(x)),$$

then in view of (5) and 4.6 congruence (8) implies that

$$A_{tp^\lambda}(x) = \prod_{k=0}^{t-1} \prod_{\tau=0}^{p^\lambda-1} f_2'\left(f^\tau(x) + kp^\lambda \xi(x) \prod_{j=0}^{\tau-1} f_2'(f^j(x))\right) \pmod{p^2},$$

or, applying 4.6,

$$\begin{aligned} A_{tp^\lambda}(x) &= \prod_{k=0}^{t-1} \prod_{\tau=0}^{p^\lambda-1} \left(f_2'(f^\tau(x)) + 2kp\xi(x)\theta(f^\tau(x)) \prod_{j=0}^{\tau-1} f_2'(f^j(x))\right) \\ &\equiv \prod_{k=0}^{t-1} \Bigg(\prod_{\tau=0}^{p^\lambda-1} f_2'(f^\tau(x)) \\ &\qquad + 2kp\xi(x) \sum_{s=0}^{p^\lambda-1} \theta(f^s(x)) \frac{\prod_{j=0}^{p^\lambda-1} f_2'(f^j(x))}{f_2'(f^s(x))} \prod_{j=0}^{s-1} f_2'(f^j(x)) \Bigg) \pmod{p^2}. \end{aligned} \tag{9}$$

According to (6),

$$\prod_{j=0}^{p^\lambda-1} f_2'(f^j(x)) = 1 + p\epsilon$$

for a suitable $\epsilon \in \mathbb{Z}_p$; consequently, (9) implies that

$$A_{tp^\lambda}(x) \equiv \prod_{k=0}^{t-1}\left(1 + p\epsilon + 2kp\xi(x)\sum_{s=0}^{p^\lambda-1}\theta(f^s(x))\frac{\prod_{j=0}^{s-1} f_2'(f^j(x))}{f_2'(f^s(x))}\right)\equiv$$

$$\equiv 1 + tp\epsilon + 2p\xi(x)\left(\sum_{k=0}^{t-1} k\right)\cdot\left(\sum_{s=0}^{p^\lambda-1}\theta(f^s(x))\frac{\prod_{j=0}^{s-1} f_2'(f^j(x))}{f_2'(f^s(x))^2}\right)\equiv$$

$$\equiv 1 + tp\epsilon + pt(t-1)\xi(x)\sum_{s=0}^{p^\lambda-1}\theta(f^s(x))\frac{\prod_{j=0}^{s-1} f_2'(f^j(x))}{f_2'(f^s(x))^2} \pmod{p^2}. \qquad (10)$$

Now combining together (2), (3), (7), and (10) we conclude that

$$f^{(t+1)p^\lambda}(x) \equiv f^{tp^\lambda+p^\lambda}(x)$$

$$\equiv f^{tp^\lambda}(x) + p^\lambda\xi(x) + \epsilon tp^{\lambda+1}\xi(x) + p^{\lambda+1}t^2\xi(x)^2 B_{p^\lambda}(x) \pmod{p^{\lambda+2}}. \qquad (11)$$

Finally, combining (11) and (2) with obvious induction on n we obtain that

$$f^{np^\lambda}(x) \equiv x + np^\lambda\xi(x) + \epsilon p^{\lambda+1}\xi(x)\frac{n(n-1)}{2}$$

$$+ p^{\lambda+1}\xi(x)^2 B_{p^\lambda}(x)\frac{n(n-1)(2n-1)}{6} \pmod{p^{\lambda+2}}$$

or, in particular,

$$f^{p^{\lambda+1}}(x) \equiv x + p^{\lambda+1}\xi(x) \pmod{p^{\lambda+2}},$$

since $p \neq 2, 3$. But the latter congruence in view of (2) implies that

$$f^{p^{\lambda+1}}(x) \not\equiv x \pmod{p^{\lambda+2}}.$$

This finally proves theorem 4.1 $\square$

Note. With the use of theorem 4.1 we can determine whether the given integer-valued and compatible polynomial $f(x) \in \mathbb{Q}_p[x]$ is ergodic. Represent $f(x)$ in the form $f(x) = \frac{g(x)}{r}$, where $r \in \mathbb{Z}_p$ and $g(x) \in \mathbb{Z}_p[x]$ and at least one coefficient of $g(x)$ is coprime with p. In fact, we can take r to be a common denominator of all coefficients of $f(x)$ represented as irreducible fractions. Here we assume that $f(x)$ is represented in the basis $(x)_0 = 1, (x)_1 = x, (x)_2 = x(x-1), \ldots$ of descending factorial powers, or in a standard basis $1, x, x^2, \ldots$. Then $\rho(f) = \operatorname{ord}_p r$, and $\rho(f)$ does not depend on the choice of the basis. We recall that $p^{\operatorname{ord}_p r}$ is the greatest power of p which is a factor of r. Now we easily find $\lambda(f)$ and determine whether f is transitive on $\mathbb{Z}/p^{\lambda(f)+1}$ (e.g., by direct calculations). In view of 4.1 for $p \neq 2, 3$ this is equivalent to the ergodicity of $f(x)$ (for $p = 3$ one should study transitivity of f on $\mathbb{Z}/p^{\lambda(f)+2}$).

Moreover, it is possible for each prime p to determine whether a polynomial $f(x) \in \mathbb{Q}_p[x]$ is integer-valued, compatible, and ergodic, calculating its values at $O(\deg f)$ points. Namely, the following is true.

4.7 Proposition. *A polynomial $f(x) \in \mathbb{Q}_p[x]$ is integer-valued, compatible and ergodic iff the mapping*

$$z \mapsto f(z) \bmod p^{\lfloor \log_p(\deg f)\rfloor+3},$$

with z ranging over $\{0, 1, \dots, p^{\lfloor \log_p(\deg f)\rfloor+3}-1\}$, defines a compatible and transitive function on the residue class ring $\mathbb{Z}/p^{\lfloor \log_p(\deg f)\rfloor+3}$.

Proof. Coefficients $a_i \in \mathbb{Q}_p$ $(i = 0, 1, \dots, d)$ of the polynomial $f(x)$ of degree d, which is represented in the form $f(x) = \sum_{i=0}^{d} a_i \binom{x}{i}$ (see $(\diamondsuit)$ of section 2), are defined by the values this polynomial $f(x)$ takes at the points $0, 1, \dots, d$. In other words, all values $f(0), f(1), \dots, f(d)$ are p-adic integers iff all coefficients $a_i \in \mathbb{Q}_p$ $(i = 0, 1, \dots, d)$ are p-adic integers, i.e., iff the polynomial $f(x)$ is integer-valued (see the beginning of section 2). Similarly, in view of theorem 2.1, the polynomial $f(x)$ preserves all congruences of the ring $\mathbb{Z}/p^{\lfloor \log_p d\rfloor+1}$ iff $\|a_i\| \le p^{-\lfloor \log_p i\rfloor}$ for all $i = 1, 2, \dots, d$, i.e., iff $f(x)$ is compatible on $\mathbb{Z}_p$. In other words, to determine whether a polynomial $f(x)$ is integer-valued and compatible it is sufficient (and necessary) to determine whether it induces a compatible function on the ring $\mathbb{Z}/p^k$ for some (arbitrarily fixed) $k \ge \lfloor \log_p d\rfloor + 1$.

In force of theorem 4.1, for $p \ne 2$, an integer-valued and compatible polynomial $f(x)$ is ergodic iff it is transitive modulo p^k for any arbitrarily fixed $k \ge \lambda(f) + 2$. Representing $f(x)$ as $f(x) = b_0 + \sum_{i=1}^{d} b_i p^{\lfloor \log_p i\rfloor} \binom{x}{i}$, $b_j \in \mathbb{Z}_p$ for $j = 0, 1, 2, \dots$, we conclude that $\rho(f)$ is the least nonnegative integer rational that is not smaller than any of $\operatorname{ord}_p i! - \lfloor \log_p i\rfloor - \operatorname{ord}_p b_i$ $(i = 1, 2, \dots, d)$. Thus, since the function $\operatorname{ord}_p i! - \lfloor \log_p i\rfloor$ does not decrease (see the proof of lemma 4.2), each $k \in \mathbb{N}$ that satisfies the inequality $2\frac{p^k-1}{p-1} - k > \operatorname{ord}_p d! - \lfloor \log_p d\rfloor$ can not be smaller than $\lambda(f)$. Yet $\operatorname{ord}_p d! = \frac{1}{p-1}(d - \operatorname{wt}_p d)$ (where, we recall, $\operatorname{wt}_p d$ is a sum of all digits in the base-p expansion of d); so taking arbitrary $k \in \mathbb{N}$ that satisfies the inequality

$$2\frac{p^k - 1}{p - 1} - k > \frac{d}{p - 1}, \tag{1}$$

we obtain that $k \ge \lambda(f)$. Elementary considerations, however, show that $k = \lfloor \log_p d\rfloor + 1$ satisfies inequality (1), thus proving the proposition for $p \ne 2$.

For $p = 2$ a polynomial $f(x) \in \mathbb{Q}_2[x]$ of degree d is integer-valued, compatible, and ergodic iff it is of the form

$$f(x) = 1 + x + \sum_{i=0}^{d} b_i \, 2^{\lfloor \log_2(i+1)\rfloor+1} \binom{x}{i}, \tag{2}$$

where $b_i \in \mathbb{Z}_2$, $i = 0, 1, 2, \dots, d$ (see theorem 2.3). Since coefficients of the polynomial $f(x)$ in its representation in the base $\binom{x}{i}$, $i = 0, 1, 2, \dots$, are uniquelly determined by the values $f(z)$ at the points $z = 0, 1, \dots, d$, to verify whether the polynomial $f(x)$ satisfies conditions (2) it is sufficient to calculate its values at the points $z = 0, 1, \dots, 2^r - 1$, where $r \in \mathbb{N}$ is an arbitrarily fixed number such that $d \le 2^r - 1$. So one can take, for instance, $r = \lfloor \log_2(d+1)\rfloor + 1$, or $r = \lfloor \log_2 d\rfloor + 3$. This finishes the proof of 4.7. $\square$

Note. Proposition 4.4 shows that for $p \neq 2$ any function $f \in \mathcal{A}$ satisfies assumptions of proposition 3.9; hence, since $N_1(f) \leq N_2(f)$, the function f preserves measure iff it is bijective modulo $p^{\lambda(f)+2}$. By the argument similar to those of the proof of proposition 4.7, one could prove the following

4.8 Proposition. *A polynomial $f(x) \in \mathbb{Q}_p[x]$ is integer-valued, compatible and measure-preserving iff the mapping*

$$z \mapsto f(z) \bmod p^{K_f},$$

with $K_f = \lfloor \log_p(\deg f) \rfloor + 3$ and z ranging over $0, 1, \dots, p^{K_f} - 1$, induces a compatible and bijective function on the ring $\mathbb{Z}/p^{K_f}$. $\square$

Again, the bounds for $\zeta(f)$ and $\eta(f)$ we mentioned at the beginning of the section cold be sharpened for various important proper subclasses of $\mathcal{A}$ in comparison with those given by theorem 4.1 and by propositions 4.7 – 4.8. The case when a function is analytic on $\mathbb{Z}_p$ (i.e., when it can be represented by power series that converges everywhere on $\mathbb{Z}_p$) seems to be of particular importance.

It is well known (see e.g. [3, Ch. 14. Section 4]) that power series $\sum_{i=0}^{\infty} c_i x^i$ ($c_i \in \mathbb{Q}_p,\ i = 0, 1, 2 \dots$) converges everywhere on $\mathbb{Z}_p$ iff $\lim\limits_{i\to\infty}\limits^{p} c_i = 0$; under the latter condition the series defines a continuous function on $\mathbb{Z}_p$. Of course, in general this function may not be integer-valued, not speaking about compatibility. Consider, however, a special case when all coefficients c_i are p-adic integers. Namely, in the ring $\mathbb{Z}_p[[x]]$ of all formal power series in one variable x over the ring $\mathbb{Z}_p$ consider a set $\mathcal{C}(x)$ of all series

$$s(x) = \sum_{i=0}^{\infty} c_i x^i \quad (c_i \in \mathbb{Z}_p,\ i = 0, 1, 2 \dots),$$

that converges everywhere on $\mathbb{Z}_p$. In other words, $s(x) \in \mathcal{C}(x)$ iff $\lim\limits_{i\to\infty}\limits^{p} c_i = 0$. Under these assumptions the series $s(x) \in \mathcal{C}(x)$ defines on $\mathbb{Z}_p$ an integer-valued function $s : \mathbb{Z}_p \to \mathbb{Z}_p$. It turnes out that this function s is uniformly differentiable and has integer-valued derivative everywhere on $\mathbb{Z}_p$.

Consider a formal derivative $s'(x) \in \mathbb{Z}_p[[x]]$ of the series $s(x)$:

$$s'(x) = \sum_{i=1}^{\infty} i c_i x^{i-1}.$$

Since $0 \leq \|ic_i\|_p = \|i\|_p \|c_i\|_p \leq \|c_i\|_p$, and $\lim\limits_{i\to\infty}\limits^{p} c_i = 0$, we conclude that $\lim\limits_{i\to\infty}\limits^{p} ic_i = 0$, and hence that $s'(x) \in \mathcal{C}(x)$. We assert that the function $s' : \mathbb{Z}_p \to \mathbb{Z}_p$ is a derivative of the function $s : \mathbb{Z}_p \to \mathbb{Z}_p$ with respect to the p-adic distance.

Indeed, it is known that in the ring $\mathbb{Z}_p[[x, y]]$ of all formal power series in variables x, y over $\mathbb{Z}_p$ the following equality holds:

$$s(x + y) = \sum_{i=0}^{\infty} \frac{s^{(i)}(x)}{i!} y^i,$$

where $s^{(i)}(x) \in \mathbb{Z}_p[[x]]$ $(i = 1, 2, \dots)$ is the ith formal derivative of the series $s(x)$, and $s^{(0)}(x) = s(x)$. By the assertion proven above, $s^{(i)}(x) \in \mathcal{C}(x)$ for all $i = 0, 1, 2, \dots$. Thus,

$$\frac{s^{(i)}(u)}{i!} = \sum_{j=i}^{\infty} c_j \binom{j}{i} u^{j-i} \in \mathbb{Z}_p$$

for each $u \in \mathbb{Z}_p$. But

$$\left\| \frac{s^{(i)}(u)}{i!} \right\|_p = \left\| \sum_{j=i}^{\infty} c_j \binom{j}{i} u^{j-i} \right\|_p \leqslant \max\{\|c_j\|_p : j = i, i+1, \dots\},$$

and consequently,

$$\lim_{i \to \infty}^{p} \frac{s^{(i)}(u)}{i!} = 0,$$

since $\lim\limits_{i \to \infty}^{p} c_i = 0$. Thus, for each $u \in \mathbb{Z}_p$ we have that

$$s(u + y) = \sum_{i=0}^{\infty} \frac{s^{(i)}(u)}{i!} y^i \in \mathcal{C}(y). \qquad (\spadesuit)$$

Finally, if $s(x) \in \mathcal{C}(x)$, then Taylor's series $(\spadesuit)$ at the point $u \in \mathbb{Z}_p$ converges to s everywhere on $\mathbb{Z}_p$. In particular, for $h \in \mathbb{Z}_p$ we obtain

$$s(u + h) = s(u) + s'(u)h + \alpha(u, h),$$

where $\lim\limits_{h \to 0}^{p} \dfrac{\alpha(u,h)}{h} = \lim\limits_{h \to 0}^{p} h \sum_{i=2}^{\infty} \frac{s^{(i)}(u)}{i!} h^{i-2} = 0$, since $\sum_{i=2}^{\infty} \frac{s^{(i)}(u)}{i!} h^{i-2} \in \mathbb{Z}_p$ in view of the equality $\lim\limits_{i \to \infty}^{p} \frac{s^{(i)}(u)}{i!} = 0$, which just has been proven above. So, $s'(u)$ is the derivative of the function s at the point u. Thus, the set $\mathcal{C}(x)$ is closed with respect to differentiations, and all the functions that are defined by series of $\mathcal{C}(x)$ are infinitely many times differentiable.

Further, let

$$s(x) = \sum_{i=0}^{\infty} s_i \binom{x}{i}$$

be an interpolation series for the function $s(x) \in \mathcal{C}(x)$. We assert that $\dfrac{s_i}{i!}$ is p-adic integer for all $i = 0, 1, 2, \dots$. Actually,

$$s(x) = \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k \sum_{i=0}^{k} S_2(k, i) i! \binom{x}{i} = \sum_{i=0}^{\infty} i! \binom{x}{i} \sum_{k=i}^{\infty} S_2(k, i) c_k,$$

where $S_2(k, i)$ is a Stirling's number. Since $\lim\limits_{i \to \infty}^{p} c_i = 0$, then $\lim\limits_{k \to \infty}^{p} S_2(k, i) c_k = 0$, because all Stirling's numbers $S_2(k, i)$ are integer rationals, i.e., $\|S_2(k, i)\|_p \leq 1$.

Consequently, the series $\sum_{k=i}^{\infty} S_2(k,i)c_k$ converges to some $A_i \in \mathbb{Z}_p$ for all $i = 0,1,2,\dots$. This proves our assertion since

$$s_i = i!A_i \quad (i = 0,1,2,\dots). \tag{$\bigstar$}$$

Put

$$\mathcal{B}(x) = \left\{ f(x) = \sum_{i=0}^{\infty} a_i \binom{x}{i} : \frac{a_i}{i!} \in \mathbb{Z}_p, \quad i = 0,1,2,\dots \right\}.$$

In other words, $\mathcal{B}(x)$ is a ring of all formal descending factorial power series over $\mathbb{Z}_p$. Each series $f(x) \in \mathcal{B}(x)$ defines on $\mathbb{Z}_p$ an integer-valued and uniformly continuos function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ (see the beginning of the section 2). This function f is compatible in view of 2.1, since we have shown during the proof of lemma 4.2 that $\operatorname{ord}_p(i!) - \lfloor \log_p i \rfloor$ is non-negative and non-descending function on $\mathbb{N}_0$. Denote by $\mathcal{B}$ (respectively, by $\mathcal{C}$) a class of all functions defined by all series of $\mathcal{B}(x)$ (respectively, of $\mathcal{C}(x)$). Obviously, $\mathcal{B}(x)$, $\mathcal{B}$, $\mathcal{C}(x)$, $\mathcal{C}$ are rings.

Further, any two distinct series of $\mathcal{B}(x)$ (respectively, of $\mathcal{C}(x)$) define two distinct functions on $\mathbb{Z}_p$: For the series of $\mathcal{B}(x)$ see the beginning of the section 2. As for the series of $\mathcal{C}(x)$, note that the above mentioned interpolation series for $s(x) \in \mathcal{C}(x)$ defines a function that is identically 0 on $\mathbb{Z}_p$ iff all its coefficients s_i are 0 (hence, $A_i = 0$, $i = 0,1,2,\dots$), see ($\bigstar$). Yet $A_i = \sum_{k=i}^{\infty} S_2(k,i)c_k$, hence $c_i = \sum_{k=i}^{\infty} S_1(k,i)A_k = 0$, where $S_1(k,i), S_2(k,i)$ are Stirling's numbers of respective kind, and the assertion follows. Thus, the rings $\mathcal{B}(x)$ and $\mathcal{B}$ (respectively, $\mathcal{C}(x)$ and $\mathcal{C}$) are isomorphic one to another; so further we do not differ series from the function it defines.

Note also that the incluion $\mathcal{B} \supset \mathcal{C}$ (see ($\bigstar$)) is strict. Obviously, $f(x) = \sum_{i=0}^{\infty} (x)_i \in \mathcal{B}$, since $f(x) = \sum_{i=0}^{\infty} i!\binom{x}{i}$. Yet $f(x) \notin \mathcal{C}$. Moreover, this function is not even analytic on $\mathbb{Z}_p$: According to [3, Ch.4, Theorem 4] a function represented by the interpolation series ($\diamondsuit$) of the section 2 is analytic on $\mathbb{Z}_p$ iff $\lim\limits_{i\to\infty}\limits^{p} \frac{a_i}{i!} = 0$.

So, a function of $\mathcal{B}$ (in a contrast to one of $\mathcal{C}$), generally speaking, can not be represented by Taylor's series that converges everywhere on $\mathbb{Z}_p$. Newertheless, all functions of $\mathcal{B}$ are differentiable at all points of $\mathbb{Z}_p$, and $\mathcal{B}$ is closed with respect to derivations: If $f \in \mathcal{B}$, then $f' \in \mathcal{B}$.

To prove the latter assertion recall that a uniformly continuous on $\mathbb{Z}_p$ function f that is represented by the interpolation series ($\diamondsuit$) is differentiable everywhere on $\mathbb{Z}_p$ iff

$$\lim_{i\to\infty}\limits^{p} \frac{a_{i+n}}{i} = 0 \tag{$\blacklozenge$}$$

for all $n \in \mathbb{N}_0$ (see [3, Ch. 13, Theorem 2]). The lattter condition obviously holds for $f \in \mathcal{B}$, since $\operatorname{ord}_p a_i \geqslant \operatorname{ord}_p(i!) = \frac{1}{p-1}(i - \operatorname{wt}_p i)$, and $\lfloor \log_p i \rfloor \geqslant \operatorname{ord}_p i$ for all $i = 0,1,2,\dots$. Thus, the derivative f' of the function f is defined everywhere on $\mathbb{Z}_p$, and

$$f'(x) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{\Delta^i f(x)}{i},$$

in case this series converges. Yet $\frac{\Delta^i f(x)}{i} = \frac{1}{i}\sum_{j=i}^{\infty} a_j \binom{x}{j-i}$, consequently,

$$\sum_{i=1}^{\infty}(-1)^{i+1}\frac{\Delta^i f(x)}{i} = \sum_{k=0}^{\infty}\binom{x}{k}\sum_{i=1}^{\infty}(-1)^{i+1}\frac{a_{k+i}}{i}.$$

But in view of (♦) the series $\sum_{i=1}^{\infty}(-1)^{i+1}\frac{a_{k+i}}{i}$ converges for each $k \in \mathbb{N}_0$ to a certain $S_k \in \mathbb{Q}_p$, and $\operatorname{ord}_p \frac{a_{k+i}}{i} = \operatorname{ord}_p a_{k+i} - \operatorname{ord}_p i \geq \operatorname{ord}_p((k+i)!) - \lfloor \log_p i \rfloor = \frac{1}{p-1}(i + k - \operatorname{wt}_p(i+k)) - \lfloor \log_p i \rfloor = \frac{1}{p-1}(i - \operatorname{wt}_p i) - \lfloor \log_p i \rfloor + \frac{1}{p-1}(k - \operatorname{wt}_p k) + \frac{1}{p-1}(\operatorname{wt}_p k - \operatorname{wt}_p(i+k) + \operatorname{wt}_p i) \geq \frac{1}{p-1}(k - \operatorname{wt}_p k) = \operatorname{ord}_p(k!)$. (The latter inequality holds since $\frac{1}{p-1}(i - \operatorname{wt}_p i) \geq \lfloor \log_p i \rfloor$ and $\frac{1}{p-1}(\operatorname{wt}_p k - \operatorname{wt}_p(i+k) + \operatorname{wt}_p i) = \operatorname{ord}_p \binom{i+k}{i} \geq 0$)). Thus, $\dfrac{S_k}{k!} \in \mathbb{Z}_p$ for all $k \in \mathbb{N}_0$; hence $f' \in \mathcal{B}$.

With the use of these results we are able to prove now the following

4.9 Theorem. *A function $f \in \mathcal{B}$ preserves measure iff it is bijective modulo p^2. The function f is ergodic iff it is transitive modulo p^2 (for $p \neq 2, 3$), or modulo p^3 (for $p \in \{2, 3\}$).*

Proof. The definition of $\mathcal{B}$ immediately implies that $\rho(f) = 0$ for each $f \in \mathcal{B}$, hence, $\lambda(f) = 1$. Thus, for $p \neq 2$ the second assertion of the theorem follows from 4.1.

To prove the first assertion, in view of 3.9 it is sufficient to demonstrate that f is uniformly differentiable modulo p, and $N_1(f) \leq 1$; that is

$$f(z + p^k r) \equiv f(z) + p^k r f'(z) \pmod{p^{k+1}} \tag{1}$$

for all $z, r \in \mathbb{Z}_p$ and $k = 1, 2, \ldots$. Since $f, f' \in \mathcal{B}$, these both functions are compatible, so it is sufficient to prove (1) for $z, r \in \mathbb{N}_0$. Since for $r = 0$ congruence (1) is trivial, we may additionally assume that $p^k r = n \in \mathbb{N}$.

Further, since $\frac{f(z+n)-f(z)}{n} = \sum_{i=1}^{\infty}\binom{n-1}{i-1}\frac{\Delta^i f(x)}{i}$, $f'(z) = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{\Delta^i f(z)}{i}$, to prove (1) it is sufficient to show that

$$\sum_{i=1}^{\infty}\left(\binom{n-1}{i-1} - (-1)^{i+1}\right)\frac{\Delta^i f(z)}{i} \equiv 0 \pmod{p}. \tag{2}$$

Yet $\frac{\Delta^i f(x)}{i} = \frac{1}{i}\sum_{j=i}^{\infty} a_j \binom{x}{j-i}$, thus, in view of 4.2, for $p \neq 2$ there holds a congruence $\frac{\Delta^i f(x)}{i} \equiv 0 \pmod{p}$ for all $i \geq 2p$. So within this case (2) is equivalent to the congruence

$$\sum_{i=1}^{2p-1}\left(\binom{n-1}{i-1} - (-1)^{i+1}\right)\frac{\Delta^i f(z)}{i} \equiv 0 \pmod{p}. \tag{3}$$

Since f is compatible, the congruence $\frac{\Delta^i f(x)}{i} \equiv 0 \pmod{p}$ may not hold only for, might be, $i = sp^m$, ($m \in \mathbb{N}_0$, $s \in \{1, 2, \ldots, p-1\}$), see [11, lemma 3.4]. Now, since $n = p^k r$, (3) immediately follows from the already mentioned Lucas' theorem, thus proving the first assertion of 4.9 for $p \neq 2$.

Now, if $p = 2$, then (2) is equivalent to

$$\sum_{i=0}^{\infty}\left(\binom{2^k r-1}{2^i-1}+1\right)\frac{\Delta^{2^i} f(z)}{2^i} \equiv 0 \pmod 2. \tag{4}$$

Yet since $\frac{a_j}{j!} \in \mathbb{Z}_2$ for all $j = 0, 1, 2, \ldots$, we conclude that $\operatorname{ord}_2 a_{2^i+m} \geq \operatorname{ord}_2 (2^i)! = 2^i - 1$ for all $m = 0, 1, 2, \ldots$; consequently, $\frac{\Delta^{2^i} f(z)}{2^i} \not\equiv 0 \pmod 2$ only for, might be, $i = 0$, thus proving (4).

Finally, the rest part of the assertion of theorem 4.9 for $p = 2$ follows from 2.3: As $\operatorname{ord}_2 i! \leq \operatorname{ord}_2 a_i$ for all $i = 0, 1, 2, \ldots$, and $\operatorname{ord}_2 i! = i - \operatorname{wt}_2(i)$, one by an elementary argument could show that $\lfloor \log_2(i+1) \rfloor + 1 \leq i - \operatorname{wt}_2(i) \leq \operatorname{ord}_2 a_i$ for $i \geq 4$; and $\operatorname{ord}_2 a_i \geq 3$. This implies that necessary and sufficient conditions of ergodicity of a function defined by interpolation series ($\diamondsuit$) of section 2 hold for all coefficients a_i with $i \geq 4$. These conditions for the rest of the coefficents are equivalent to the transitivity of f modulo 8, since $a_i \equiv 0 \pmod 8$ for $i \geqslant 4$. $\square$

Note. Theorem 4.9 demonstrates that sufficient and necessary conditions of transitivity modulo p^n for polynomials with integer rational coefficients established by M. V. Larin in [15] remain valid for a wider class (namely, for $\mathcal{B}$) of functions. It turnes out, however, that all these functions modulo each p^n could be expressed as polynomials with rational integer coefficients.

Namely, from the definition of the class $\mathcal{B}$ it easily follows that each function $f \in \mathcal{B}$ could be uniformly approximated by polynomials over $\mathbb{Z}_p$: For each $n \in \mathbb{N}$ there exists a polynomial $f_n(x) \in \mathbb{Z}_p[x]$ such that $f(z) \equiv f_n(z) \pmod{p^n}$ for all $z \in \mathbb{Z}_p$. Actually, the series $\sum_{j=0}^{\infty} r_j \binom{x}{j}$ defines a function that is identically 0 modulo p^n iff all $r_j \equiv 0 \pmod{p^n}$ (see [11, proposition 4.2]). So we may put $f_n(x) = \sum_{i=0}^{\omega(n)} a_i \binom{x}{i}$, where $\omega(n) = \max\{j \in \mathbb{N}_0 \colon \frac{1}{p-1}(j - \operatorname{wt}_p j) < n\}$.

It turnes out that the inverse assertion is also true: Suppose a function $f \colon \mathbb{Z}_p \to \mathbb{Z}_p$ could be uniformly approximated by polynomials over $\mathbb{Z}_p$ in the above mentioned sence; then $f \in \mathcal{B}$. To prove this assertion assume that $f(z) \equiv f_i(z) \pmod{p^i}$ for all $z \in \mathbb{Z}_p$, where $f_i(x) \in \mathbb{Z}_p[x]$, $i = 1, 2, \ldots$. Each polynomial $f_i(x)$ of degree d_i admits one and the only representation in the form ($\diamondsuit$) of section 2: $f_i(x) = \sum_{j=0}^{d_i} a_{ij} \binom{x}{j}$, where $a_{ij} \in \mathbb{Z}_p$ and $\operatorname{ord}_p a_{ij} \geq \operatorname{ord}_p (j!)$ in view of ($\bigstar$), since $f_i \in \mathcal{C} \subset \mathcal{B}$. For the given function f each polynomial $f_i(x)$ is uniquelly determined up to a summand that is 0 modulo p^i everywhere on $\mathbb{Z}_p$. So we may assume that $d_i = \omega(i)$ (see above); then coefficients of the polynomial $f_i(x)$ are defined uniquelly up to the summands with p-adic norms not exceeding p^{-i}. This implies that $a_{i+1,j} \equiv a_{ij} \pmod{p^i}$ (we assume $a_{ij} = 0$ for $j > \omega(i)$). Hence, $\lim\limits_{i \to \infty}\limits^{p} a_{ij} = a_j \in \mathbb{Z}_p$, and $\frac{a_j}{j!} \in \mathbb{Z}_p$. Consequently, the series $\sum_{i=0}^{\infty} a_i \binom{x}{i}$ defines a uniformly continuous on $\mathbb{Z}_p$ function $\tilde{f} \in \mathcal{B}$; the latter must be equal to f since $f(z) \equiv f_i(z) \equiv \tilde{f}(z) \pmod{p^i}$ for all $z \in \mathbb{Z}_p$ and all $i = 1, 2, \ldots$.

Now we define a non-Archimedian pseudo-valuation on $\mathcal{B}$ as $\max\{\|f(z)\|_p \colon z \in \mathbb{Z}_p\}$ for $f \in \mathcal{B}$. The just proven results imply that with respect to the distance D_p, induced by this pseudo-valuation, the ring $\mathcal{B}$ is a complete metric space; actually, $\mathcal{B}$ is a completion with respect to D_p of the space $\mathcal{P} \subset \mathcal{C}$ of all functions induced on $\mathbb{Z}_p$ by polynomials over $\mathbb{Z}$ (in particular, the space $\mathcal{B}$ is separable).

This implies, in turn, that $\mathcal{B}$ (contrasting to $\mathcal{C}$) is closed with respect to composition of functions: If $f, g \in \mathcal{B}$ then $f(g) \in \mathcal{B}$. In fact, let g be uniformly approximated by the sequence $\{g_n(x) \in \mathbb{Z}_p[x] : n = 1, 2, \dots\}$, that is, $g_n(z) \equiv g(z) \pmod{p^n}$ for all $z \in \mathbb{Z}_p$. The compatibility of the function f imples then that $D_p(f(g), f(g_n)) \leq p^{-n}$, i.e., for $n \to \infty$ the sequence $f(g_n)$ tends to $f(g)$ with respect to the distance D_p. But $f(g_n) \in \mathcal{B}$ for each $n = 1, 2, \dots$: If f is uniformly approximated by the sequence $\{f_m(x) \in \mathbb{Z}_p[x] : m = 1, 2, \dots\}$, then $f_m(g_n(z)) \equiv f(g_n(z)) \pmod{p^m}$ for all $z \in \mathbb{Z}_p$. Hence, the sequence $\{f_m(g_n(x)) \in \mathbb{Z}_p[x] : m = 1, 2, \dots\}$ tends to the function $f(g_n)$ with respect to the distance D_p, and $f_m(g_n) \in \mathcal{B}$, since it is a polynomial over $\mathbb{Z}_p$. Consequently, $f(g) \in \mathcal{B}$ in view of completeness of $\mathcal{B}$.

Thus, we have proven the following

4.10 Proposition. *The ring $\mathcal{B}$ is a separable and complete with respect to the distance D_p metric space of functions that are differentiable everywhere on $\mathbb{Z}_p$. $\mathcal{B}$ is closed with respect to compositions of functions and with respect to derivations. The countable set $\mathcal{P}$ of all polynomials over $\mathbb{Z}$ is a dence subset of $\mathcal{B}$.* $\square$

To make use of criterion 4.9 for applications to pseudorandom number generation it is important to have a number of examples of functions of $\mathcal{B}$ that could be more or less easily implemented as computer programs. As we have mentioned above, all polynomials over $\mathbb{Z}_p$ are in $\mathcal{B}$.

Functions that are rational over $\mathbb{Z}_p$, that is, those of the form $f(x) = \frac{u(x)}{v(x)}$, where $u(x), v(x) \in \mathbb{Z}_p[x]$, are also in $\mathcal{B}$, providing the denominator vanishes modulo p nowhere on $\mathbb{Z}_p$ (in view of compatibility it is sufficient to verify the latter condition only for the points of $\{0, 1, \dots, p-1\}$). Indeed, for each $z \in \mathbb{Z}_p$ the element $v(z)$ is not 0 modulo p, and hence has a multiplicative inverse in the ring $\mathbb{Z}/p^n$. Thus $\frac{u(z)}{v(z)} \equiv u(z)v(z)^{\phi(p^n)-1} \pmod{p^n}$, where ϕ is Euler's totient function. Hence, the function f could be uniformly approximated by polynomials $u(x)v(x)^{\phi(p^n)-1} \in \mathbb{Z}_p[x]$, $n = 1, 2, \dots$; so $f \in \mathcal{B}$ in force of 4.10.

Another type of functions of $\mathcal{B}$ are exponential ones. For instance, consider a function a^x with $a \equiv 1 \pmod{p}$ (hence, $a = 1 + pr$ for a suitable $r \in \mathbb{Z}_p$). Then $a^x = \sum_{i=0}^{\infty} p^i r^i \binom{x}{i}$; it is well known (see e.g. [3, Ch. 14, Section 5]) that for $p \neq 2$ this function is analytic on $\mathbb{Z}_p$ (hence, lies in $\mathcal{C}$). If $p = 2$ and r is odd, then a^x is not analytic on $\mathbb{Z}_2$, thus not in $\mathcal{C}$. Newertheles, within the latter case a^x is in $\mathcal{B}$, since $\mathrm{ord}_2\,(i!) = i - wt_2\, i$ and hence $(1+2r)^x = \sum_{i=0}^{\infty} 2^i r^i \binom{x}{i} \in \mathcal{B}$. It is not difficult to see that the function $(1 + 4r)^x$ is in $\mathcal{C}$. So, summing all these considerations we conclude that if $a \in \mathbb{Z}_p$, $a \equiv 1 \pmod{p}$ then the function a^x is in $\mathcal{B}$.

Exponential functions of the considered type are special cases of functions of more general form u^v, where $u(z) \equiv 1 \pmod{p}$ for all $z \in \mathbb{Z}_p$.

4.11 Lemma. *Let $u, v : \mathbb{Z}_p \to \mathbb{Z}_p$ be compatible functions and let $u(z) \equiv 1 \pmod{p}$ for all $z \in \mathbb{Z}_p$ (so it is sufficient to verify the latter condition only for $z = 0, 1, \dots, p-1$). Then the function $f(z) = u(z)^{v(z)}$ is well defined for all $z \in \mathbb{Z}_p$, integer-valued and compatible. Moreover, if $w, v \in \mathcal{B}$, $u(z) = 1 + pw(z)$, then $f \in \mathcal{B}$.*

Proof. The above considerations of functions of type a^x with $a \equiv 1 \pmod{p}$ immediately imply that the function f is well defined on $\mathbb{Z}_p$ and that it is integer-valued. To prove the compatibility of f note that for arbitrary $b, c, d \in \mathbb{Z}_p$ and

$n = 1, 2, \dots$ one has $(a + p^n b)^{c+p^n d} = (a + p^n b)^c((a + p^n b)^{p^n})^d$, since elmentary properties of powers are of the same form both in real and p-adic cases, see [3, Ch. 14, Section 5]. As both u and v are compatible functions, for arbitrary $z, r \in \mathbb{Z}_p$ there exist $s, t \in \mathbb{Z}_p$ such that $(u(z + p^n r))^{v(z+p^n r)} = (u(z) + p^n t)^{v(z)+p^n s}$; hence $(u(z+p^n r))^{v(z+p^n r)} = (u(z)+p^n t)^{v(z)}((u(z)+p^n t)^{p^n})^s \equiv (u(z)+p^n t)^{v(z)} \pmod{p^n}$ in view of the congruence $(u(z) + p^n t)^{p^n} \equiv 1 \pmod{p^n}$, which we must prove now.

As $u(z) \equiv 1 \pmod{p}$, for a suitable $k \in \mathbb{Z}_p$ we have $u(z) + p^n t = 1 + pk$. Yet $(1 + pk)^{p^n} = \sum_{i=0}^{p^n} k^i p^i \binom{p^n}{i} = \sum_{i=0}^{p^n} k^i \frac{p^i}{i!}(p^n)_i \equiv 1 \pmod{p^n}$, since $\frac{p^i}{i!} \in \mathbb{Z}_p$. Finally, denoting by $\overline{v(z)} = v(z) \bmod p^n$ the least nonnegative residue of $v(z)$ modulo p^n, for a suitable $h \in \mathbb{Z}_p$ we obtain $f(z + p^n r) \equiv (u(z) + p^n t)^{v(z)} = (u(z) + p^n t)^{\overline{v(z)}}(u(z) + p^n t)^{p^n h} \equiv (u(z) + p^n t)^{\overline{v(z)}} = \sum_{i=0}^{\overline{v(z)}} u(z)^{\overline{v(z)}-i} p^{ni} t^i \binom{\overline{v(z)}}{i} = (u(z))^{\overline{v(z)}} \equiv (u(z))^{\overline{v(z)}}(u(z))^{p^n h} = (u(z))^{v(z)}$, where $\cdot \equiv \cdot$ means $\cdot \equiv \cdot \pmod{p^n}$. Thus, f is compatible.

To prove the rest of the lemma, note that for each $z \in \mathbb{Z}_p$ and each $n = 1, 2, \dots$ the congruence $(u(z))^{v(z)} \equiv \sum_{i=0}^{n}(u(z) - 1)^i \binom{v(z)}{i} \pmod{p^n}$ holds, since $\|u(z) - 1\|_p \le \frac{1}{p}$. This implies that

(1) all functions $f_n = \sum_{i=0}^{n} \frac{p^i}{i!}(v)_i w^i$ are in $\mathcal{B}$, since all $\frac{p^i}{i!}$ are p-adic integers (see above);

(2) the sequence $\{f_n : n = 1, 2, \dots\}$ tends to f with respect to the distance D_p.

Now (1)–(2) imply that $f \in \mathcal{B}$ in force of 4.10. $\square$

With the use of these results one may construct explicit expressions for various ergodic functions that are suitable for program implementations. For instance, the following is true.

4.12 Proposition. *For $g \in \mathcal{B}$ the function $f(x) = 1 + x + p^2 g(x)$ is ergodic.*

Proof. For $p \notin \{2, 3\}$ the assertion trivially follows from 4.9. For $p \in \{2, 3\}$ in view of 4.9 it is sufficient to show that f is transitive modulo p^3. In turn, to demonstrate the latter it is sufficient to prove only that $f^{kp^2}(0) \not\equiv 0 \pmod{p^3}$ for $k = 1, 2, \dots, p - 1$, since f is transitive modulo p^2 and hence in view of the compatibility induces on $\mathbb{Z}/p^3$ a permutation such that the length of each its cycle is a multiple of p^2. Yet for all $i = 0, 1, 2 \dots$ the compatibility of g implies that $f^i(0) \equiv i + p^2 \sum_{j=0}^{i-1} g(j) \pmod{p^3}$; so $f^{kp^2}(0) \equiv kp^2 + p^2 \sum_{j=0}^{kp^2-1} g(j) \equiv kp^2 + p^2 \sum_{z=0}^{p-1} g(z)pk \equiv kp^2 \pmod{p^3}$ since (again in view of the compatibility of g) the congruence $s \equiv r \pmod{p}$ implies the congruence $p^2 g(r) \equiv p^2 g(s) \pmod{p^3}$. $\square$

5. Applications: a discussion.

The results obtained in previous sections might have applications to the design of pseudorandom number generators that have relatively simple program implementation, generate purely periodic sequences of numbers of $\{0, 1, \dots, m - 1\}$ and provide certain guarantee for the statistical quality of these sequences, their uniform distribution at the first turn. Speaking about relatively simple program implementation we mean that the considered generators depend on certain parameters that determine the performance; one may vary these parameters to achieve the desired performance without loosing the quality.

For $m = p^k$, p prime, these sequences could be generated as the first order recurrence sequences that satisfy the relation $x_{n+1} \equiv f(x_n) \pmod m$, where $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is any compatible and ergodic function of the considered in previous sections. In this case for each $k = 1, 2, \dots$ we obtain a purely periodic sequence with a period of length p^k such that each element of $\{0, 1, \dots, p^k - 1\}$ occurs at the period exactly once (in particular, the sequence is uniformly distributed).

An important indicator of statistical quality of the sequence is a distribution of $(r+1)$-tuples $\{(x_n, x_{n+1}, \dots, x_{n+r}) : n = 0, 1, 2, \dots\}$. Ideally, the sequence $\{\mathbf{u}_n = (\frac{x_n}{p^k}, \frac{x_{n+1}}{p^k}, \dots, \frac{x_{n+r}}{p^k}) : n = 0, 1, 2, \dots\}$ of points of $(r+1)$-dimensional Euclidean space should be uniformly distributed in the unit hypercube for all r. By no means this can be achieved for periodic sequences. For such sequences there exist some popular statistical tests that use certain characteristics of special families of hyperplanes, which are parallel one to another, and such that their union contain all the points corresponding to the sequences of $(r+1)$-tuples (see e.g. [2, section 3.3.4]). Note that if for some $c, c_0, \dots, c_r \in \mathbb{Z}$ the congruences

$$c + \sum_{i=0}^{r} c_i x_{n+i} \equiv 0 \pmod{p^k}, \qquad (n = 0, 1, 2, \dots) \qquad (\blacktriangle)$$

hold, then all the points $\mathbf{u}_n$ fall into the hyperplanes $h + \sum_{i=0}^{r} c_i X_i = 0$, which are parallel one to another. For linear congruential generators such families of parallel hyperplanes exist even for $r = 2$, and these families does not depend on k (see the introduction).

Note that if $(\blacktriangle)$ holds for some k, then for all $j = 1, 2, \dots$ the members of the sequence $\{x_n\}$ satisfy the relation $p^j c + \sum_{i=0}^{r} p^j c_i x_{n+i} \equiv 0 \pmod{p^{k+j}}$. The relations of the latter kind will be temporarily and loosely defined as trivial. Trivial relations always exist: For instance, choosing certain $K \in \mathbb{N}$ in view of the ergodicity of f we obtain for all $k \geq K$ trivial relations $p^{k-K} x_{n+p^K} \equiv p^{k-K} x_n \pmod{p^k}$. Speaking informally, the triviality of relations just means that their coefficients tend to 0 whereas k tends to infinity, i.e. trivial relations are those that degenerate to $0 = 0$ in $\mathbb{Z}_p$.

For an important wide class of nonlinear congruential generators we prove further that if the dimension of the mentiond parallel hyperplanes such that their union contains all points $\mathbf{u}_n$, $(n = 0, 1, 2, \dots)$, does not tend to infinity together with k, then this family of hyperplanes is defined by trivial relations.

Now we give exact statements.

5.1 Proposition. *Let $f \in \mathbb{Q}_p[x]$ be an integer-valued, compatible and ergodic polynomial of degree d over a field $\mathbb{Q}_p$ of p-adic numbers* (all these polynomials for $p = 2$ are completely characterized by theorem 2.3; for odd p see 2.4, 4.7 and a note preceding 4.7). *Let, further, r be a positive integer rational such that for each $k \in \mathbb{N}$ there exist $c, c_0, \dots, c_r \in \mathbb{Z}_p$ (not all congruent to 0 modulo p) that satisfy $(\blacktriangle)$. Then $d = 1$.*

We will need the following

5.2 Lemma. *Under the assumptions of proposition 5.1 let $c, c_0, \dots, c_r \in \mathbb{Z}_p$ be not depending on k, that is, let there exist $c, c_0, \dots, c_r \in \mathbb{Z}_p$ satisfying $(\blacktriangle)$ for all $k \in \mathbb{N}$ simultaneously. Then $d = 1$.*

Proof of lemma 5.2. As f is ergodic, $d \neq 0$. Assume that $d > 1$. Consider $w(x) = c + \sum_{i=0}^{r} c_i f^i(x)$. As $w(x)$ is a composition of integer-valued and compatible polynomials over $\mathbb{Q}_p$, $w(x) \in \mathbb{Q}_p[x]$ is integer-valued and compatible. Yet each $f^i(x)$ has degree d^i; hence, since $d > 1$, we conclude that $w(x)$, being a sum of polynomials of pairwise distinct degrees, must be a polynomial of nonzero degree.

Yet, since $x_{n+i} \equiv f^i(f^n(x_0)) \pmod{p^k}$, the assumptions of the lemma imply that $w(x_n) \equiv 0 \pmod{p^k}$ for all $n = 0, 1, 2, \ldots$. In other words, $w(z) \equiv 0 \pmod{p^k}$ for all $z \in \mathbb{Z}_p$, since x_n takes all values in $\{0, 1, \ldots, p^k - 1\}$ in view of the ergodicity of f, and $w(x)$ is compatible. The assumptions of the lemma now imply that $w(z) \equiv 0 \pmod{p^k}$ for all $z \in \mathbb{Z}_p$ and all $k = 1, 2, \ldots$. Consequently, $w(z) = 0$ for all $z \in \mathbb{Z}_p$ and hence the polynomial $w(x)$ must be 0 in the ring $\mathbb{Q}_p[x]$. A contradiction that proves the lemma. $\square$

Proof of proposition 5.1. By the assumption, for each $k \in \mathbb{N}$ the set $\mathcal{L}_k$ of all $\mathbf{c} = (c, c_0, \ldots, c_r) \in \mathbb{Z}_p^{r+2}$ such that $\|\mathbf{c}\|_p = 1$ and $c, c_0, \ldots, c_r$ satisfy (▲), is not empty. Obviously, $\mathcal{L}_1 \supset \mathcal{L}_2 \supset \ldots$ since f is compatible.

Further, we assert that each set $\mathcal{L}_k$ is closed in the topology of metric space $\mathbb{Z}_p^{r+2}$. Actually, if $\mathbf{c} \in \mathcal{L}_k$, $\mathbf{c}' \in \mathbb{Z}_p^{r+2}$, $\|\mathbf{c} - \mathbf{c}'\| \leq p^{-s}$, $s \geq k$, then $\mathbf{c}' = \mathbf{c} + p^s\mathbf{z}$ for a suitable $\mathbf{z} \in \mathbb{Z}_p^{r+2}$. Hence, $\|\mathbf{c}'\|_p = 1$ and $\mathbf{c}'$ satisfies (▲); consequently, $\mathbf{c}' \in \mathcal{L}_k$.

Now we apply to the sequence $\mathcal{L}_1 \supset \mathcal{L}_2 \supset \ldots$ the p-adic analogon of the lemma on nested closed intervals of real analysis. The analogon of this lemma holds for the topological spaces of much more general type, see e.g. the theorem in [16, Ch. 3, section 34, I]; the p-adic lemma could be easily deduced from the mentioned theorem. Thus, we conclude that the intersection of the sequence $\mathcal{L}_1 \supset \mathcal{L}_2 \supset \ldots$ is not empty. That is, there exists $\mathbf{c}'' \in \mathbb{Z}_p^{r+2}$ that satisfies the assumptions of lemma 5.2. Yet then $d = 1$. $\square$

From here we deduce the following

5.3 Theorem. *Let $f \in \mathbb{Q}_p[x]$ be an integer-valued compatible and ergodic polynomial with $\deg f > 1$, and let there exists $r \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ the linear complexity over the ring $\mathbb{Z}/p^k$ of the recurrence sequence $\{x_n\}$ defined by the recurrence relation $x_{n+1} \equiv f(x_n) \pmod{p^k}$, does not exceed r. In other words, let there exist $c^{(k)}, c_0^{(k)}, \ldots, c_r^{(k)} \in \mathbb{Z}_p$ such that*

$$c^{(k)} + \sum_{i=0}^{r} c_i^{(k)} x_{n+i} \equiv 0 \pmod{p^k} \qquad (n = 0, 1, 2, \ldots). \tag{◄}$$

Then $\overset{p}{\lim\limits_{k\to\infty}} c^{(k)} = \overset{p}{\lim\limits_{k\to\infty}} c_1^{(k)} = \ldots = \overset{p}{\lim\limits_{k\to\infty}} c_r^{(k)} = 0.$

Proof. To start with, we note that from the proofs of both lemma 5.2 and proposition 5.1 it follows that they remain true if we let k under their assumptions range over arbitrary infinite subset of $\mathbb{N}$.

Now for each $k \in \mathbb{N}$ choose (and fix) $c^{(k)}, c_0^{(k)}, c_1^{(k)}, \ldots, c_r^{(k)} \in \mathbb{Z}_p^{(r+2)}$ that satisfy (◄). Put $\mathbf{c}_k = (c^{(k)}, c_0^{(k)}, c_1^{(k)}, \ldots, c_r^{(k)}) \in \mathbb{Z}_p^{(r+2)}$. In view of 5.1 then $\|\mathbf{c}_k\|_p < 1$ for all $k \in \mathbb{N}$. Denote $\mathcal{N} = \{k \in \mathbb{N} \colon \|\mathbf{c}_k\|_p > p^{-k}\}$. In other words, $k \notin \mathcal{N}$ iff (◄) is equivalent to the congruence $0 \equiv 0 \pmod{p^k}$.

It is obvious that if $\mathcal{N}$ is finite, then the conclusion of the theorem is true. Let $\mathcal{N}$ be infinite.

For $k \in \mathcal{N}$ put $\hat{\mathbf{c}}_k = \|\mathbf{c}_k\|_p \mathbf{c}_k$ and denote by $\hat{\mathcal{N}}$ a set of all $m \in \mathbb{N}$ such that $p^k \|\mathbf{c}_k\|_p = p^m$ for a suitable $k \in \mathcal{N}$. In other words, we replace each (◀) with the equivalent system of congruences

$$\hat{c}^{(k)} + \sum_{i=0}^{r} \hat{c}_i^{(k)} x_{n+i} \equiv 0 \pmod{p^m} \qquad (n = 0, 1, 2, \dots),$$

where $(\hat{c}^{(k)}, \hat{c}_0^{(k)}, \hat{c}_1^{(k)}, \dots, \hat{c}_r^{(k)}) = \hat{\mathbf{c}}_k$, $p^m = p^k \|\mathbf{c}_k\|_p$.

If the set $\hat{\mathcal{N}}$ is finite, the conclusion of the theorem is obviously true. If $\hat{\mathcal{N}}$ is infinite, then, since $\|\hat{\mathbf{c}}_k\|_p = 1$, in view of 5.1 and the note at the beginning of the proof we conclude that $\deg f = 1$. A contradiction. □

In the statement of the theorem 5.3 we mention a notion of linear complexity of a sequence over a ring. This is commonly used (especially in cryptography) characteristic of a quality of a sequence. Lemma 5.2 in these terms asserts that the sequence $\{x_i = f(x_{i-1}) : i \in \mathbb{N}\}$ has infinite linear complexity over $\mathbb{Z}_p$, providing $f \in \mathbb{Q}_p[x]$ is integer-valued compatible ergodic polynomial of degree $d > 1$. This assertion could be slightly strengthened.

5.4 Corollary. *If $f \in \mathbb{Q}_p[x]$ is an integer-valued compatible ergodic polynomial of degree $d > 1$, then a recurrence sequence $\{x_n\}$ that satisfy recurrence relation $x_{n+1} = f(x_n)$, has infinite linear complexity over $\mathbb{Q}_p$.*

Proof. If for suitable $c, c_0, \dots, c_r \in \mathbb{Q}_p$, which are not 0 simultaneously, the equality $c + \sum_{j=0}^{r} c_j x_{n+j} = 0$ holds for all $n = 0, 1, 2, \dots$, then the equality $hc + \sum_{j=0}^{r} hc_j x_{n+j} = 0$ where $h = 1$ if $c, c_0, \dots, c_r \in \mathbb{Z}_p$, and $h = \|(c, c_0, \dots, c_r)\|_p$ otherwise, holds either. In view of the compatibility of f the conclusion now follows from 5.2. □

Note. The assumption $f \in \mathbb{Q}_p[x]$ within statements of 5.1–5.4 can not be omitted. For instance, let $p = 2$ and let

$$f(x) = 1 + x + 4(-1)^{1+x} = 1 + x + \sum_{j=0}^{\infty} (-1)^j 2^{j+2} \binom{x}{j}.$$

By the theorem 2.3, the integer-valued function f is compatible and ergodic. However, it is easy to see that the recurrence sequence $\{x_n \in \mathbb{Z}_2\}$ defined by the recurrence relation $x_{n+1} = f(x_n)$ satisfy the relation $x_{n+2} = x_n + 2$, i.e., has linear complexity 2 over $\mathbb{Z}_2$.

We should notice that in this section we use the notion of linear complexity of a sequence over a ring in a somewhat broader sence than it is commonly used. More often the linear complexity of a sequence $\{x_n\}$ of elements of a commutative ring R is understood as the smallest $r > 0$ such that there exist $c_0, \dots, c_{r-1} \in R$ that satisfy simultaneously all equations $x_{n+r} = \sum_{j=0}^{r-1} c_j x_{n+j}$ for $n = 0, 1, 2, \dots$. We, in distinction from the latter, consider non-homogeneous relations (i.e., with a nonzero constant term), as well as relations where all coefficients are zero divisors (yet not

all 0 simultaneously; in the assertion of 5.3 the latter, however, is not important). If R is a field, then both notions basically do not differ one from another: If a sequence satisfies a relation $c + \sum_{i=0}^{r} c_i x_{n+i} = 0$ with $c_r \neq 0$, then it satisfies the relation $x_{n+r+1} = c_r^{-1} c_0 x_n - \sum_{j=0}^{r-1} c_r^{-1}(c_j - c_{j+1}) x_{n+j+1}$. Our definition seems to us some more convenient for geometric interpretations, see above.

In other words, we have shown that, loosely speaking, nonlinear ergodic polynomial generators are absolutely nonlinear: The sequences they produce can not be implemented as linear recurrence sequences over $\mathbb{Q}_p$. We do not discuss here what is the impact of these results on measurments of the statistical quality of the corresponding output sequences by the above mentioned statistical tests, leaving this issue to a forthcoming paper. We only note that the results give some evidence that nonlinear congruential generators with integer-valued compatible ergodic polynomials over $\mathbb{Q}$ as state update functions in practice will pass these tests.

Properly restated analogons of 5.1–5.4 hold for composite $m = p_1^{k_1} \cdots p_s^{k_s}$, which is a product of powers of distinct primes $p_1, \ldots, p_s$, providing the transformation f preserves all congruences of the ring $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}$. In connection with congruential generators modulo a composite m we also note that one can take f to be a function, defined on the set $\mathbb{N}_0$ of all nonnegative integer rationals that takes values in $\mathbb{Z}$, preserves all congruences of the ring $\mathbb{Z}$, and that is ergodic as a function of integer p-adic variable for all $p \in \{p_1, \ldots, p_s\}$. These functions may also be constructed with the use of the results of the paper.

For instance, such functions may be found in the class

$$\mathcal{B}_0 = \left\{ \sum_{i=0}^{\infty} a_i \, (x)_i \colon a_i \in \mathbb{Z}; i = 0, 1, 2, \ldots \right\},$$

where, we recall, $(x)_i$ is ith descending factorial power of x: $(x)_0 = 1$, $(x)_i = x(x-1)\cdots(x-i+1)$ for all $i = 1, 2, \ldots$. It is obvious that $\mathcal{B}_0$ is a proper subclass of the class $\mathcal{B}$ (studied in section 4) for each prime p (the definition of $\mathcal{B}$, we recall, depends on p). Since $\mathcal{B}$ consists of functions that preserve all congruences of the ring $\mathbb{Z}_p$, each function g of $\mathcal{B}_0$ preserves all congruences of the ring $\mathbb{Z}$, that is, for each $a, b \in \mathbb{N}_0$ and each natural number $N > 1$ the congruence $a \equiv b \pmod{N}$ implies the congruence $g(a) \equiv g(b) \pmod{N}$. So as a state update function of a pseudorandom generator one may take, for instance,

$$f(x) = 1 + x + p_1^2 \cdots p_s^2 g(x) \qquad (g \in \mathcal{B}_0);$$

in view of 4.12 f is ergodic as a function of integer p_j-adic variable for all $j = 1, 2, \ldots, s$. That is, f is transitive modulo p_j^k for all $k = 1, 2, \ldots$ and for all $j = 1, 2, \ldots, s$. Thus, f is transitive modulo each $p_1^{t_1} \cdots p_s^{t_s}$ for arbitrary $t_1, \ldots, t_s \in \mathbb{N}$. In particular, f is transitive modulo m, and hence a pseudorandom number generator with the state update function f and arbitrary initial state $x_0 \in \{0, 1, \ldots, m-1\}$ produces a purely periodic sequence of period length m, and each number of $\{0, 1, \ldots, m-1\}$ occurs at the period of this sequence exactly once.

Obviously, $\mathcal{B}_0$ contains all polynomials with rational integer coefficients, so if $g(x) \in \mathbb{Z}[x]$ is a polynomial of degree $d \geq 1$, then the performance of the correspondig pseudorandom generator is equivalent to d additions and d multiplications

modulo m of integer rationals. Obviosly, $\mathcal{B}_0$ consists not only of polynomials over $\mathbb{Z}$. It is not clear, however, whether it contains other 'natural' functions that admit relatively simple program implementation.

Moreover, if m is arbitrary, it is not clear enough what functions should be considered as 'natural', and what should not. If by 'natural' functions one understands compositions of arithmetical operations (addition, subtraction, multiplication, division, raising to a positive integer power, exponentiation) then the functions of this kind could be constructed, for instance, with the use of 2.3, 2.4, and 4.9 combined with 4.11 and 4.12. So, theorems 2.3–2.4 imply that a polynomial $f(x) \in \mathbb{Q}[x]$ of a form

$$f(x) = 1 + x + \sum_{i=0}^{d} c_i \, p_1^{\lfloor \log_{p_1}(i+1) \rfloor + 1} \cdots p_s^{\lfloor \log_{p_s}(i+1) \rfloor + 1} \binom{x}{i},$$

for arbitrary $c_0, c_1, c_2 \ldots \in \mathbb{Z}$ is transitive modulo arbitrary natural number $M > 1$, which is a product of powers of $\{p_1, \ldots, p_s\}$; in particular $f(x)$ is transitive modulo m. Hence, the performance of the corresponding pseudorandom generator is equivalent to d multiplications, d additions, $d+1$ reductions some moduli and one division of integer rationals.

From the above formula it follows that, for instance, a polynomial $f(x) = 1 + x + \frac{5}{18}(x)_6$ is transitive modulo 10^k for all $k = 1, 2, \ldots$. In a similar way, with the use of 2.5 and 4.11 (or 4.9 together with 4.11) one may construct generators that use exponentiations. For instance, the function $f(x) = 1 + x + 201^x$ (or, more generally, the function $f(x) = 1 + x + (1 + 200u(x))^{w(x)}$ with $u(x), v(x) \in \mathbb{Z}[x]$), as well as the function $f(x) = 1 + x + 201^{201^x}$ are transitive modulo 10^k for all $k = 1, 2, \ldots$ (see 4.9 and 4.11); the same is true for the function $f(x) = 1 + x + 100 \cdot 11^x$ (see 2.5 and 4.11). Judging by the number of publications on inversive generators, taking a multiplicative inverse (or, generally, raising to negative powers) modulo m also should be considerd as 'natural' operations. Generators of this kind also could be constructed with the use of results of the paper: For instance, taking $w(x) = -1, v(x) = x$ in the just mentioned example one obtains a function $f(x) = 1 + x + (1 + 200x)^{-1}$; it is transitive modulo 10^k for all $k = 1, 2, \ldots$.

We note that during the past decade there were intensive studies of such pseudorandom generators, as power generator ($f(x) = x^r, r \in \mathbb{N}$), exponential generator ($f(x) = a^x$), twice exponential generator ($f(x) = a^{b^x}$) and inversive generator ($f(x) = a + bx^{-1}$ or $f(x) = (a + bx)^{-1}$). The examples of generators that are mentioned above in the section, which use compositions of arithmetical operations, including exponentiation and raising to negative power, as we see, are somewhat distinct from the ones usually studied (by summand $1 + x$, at the first turn). These distinctions practically do not worsen the performance of the corresponding programs. However, these distinctions do not allow us to apply to the generators considered in this paper the results on already studied generators. It would be very useful to study the possibility of such transfer, since in this area there are a lot of important results belonging to different authors (unfortunately, we could not present even a short survey here because of hudge number of these).

At the same time, all the generators introduced in this paper are modulo the given m equivalent to generators with recurrence relation $x_{n+1} \equiv f_m(x_n) \pmod{m}$, where $f_m(x) \in \mathbb{Q}[x]$ (this is an immediate consequence of p-adic Weierstrass theo-

rem, for the latter see e.g. [3, Ch. 10, Theorem 1]). Hence, all the results obtained in literature for the so-called polynomial congruential generators could be immediately applied to generators considered in this paper, at least under extra restriction $f_m(x) \in \mathbb{Z}[x]$.

We should note also that a number of generators studied in literature concern a specific case when m is a product of two distinct large primes. The results of the current paper are of little interest for this particular case, since with the use of these results one can construct generators that are either equivalent modulo a prime divisor p of m to a linear congruential generator, or involve some given in advance transitive modulo p polynomial of degree > 1. The latter must be constructed beforehand and then 'adjusted' to make it transitive modulo some p^s, with s satisfying assumptions of 3.14, 4.1 or 4.9. The methods of such 'adjustment' we hope to publish in one of forthcoming papers, and here we restrict ourselves with an example. For instance, using these techniques and choosing a transitive modulo 5 polynomial $1+3x^3$, it is possible to construct a polynomial $1-127x-152x^3+152x^5$, which is transitive modulo 10^k, for all $k = 1, 2, \ldots$.

So in view of these considerations, the methods of construction of pseudorandom generators we develope in this paper could give the best effect if they are applied to the case when m is a product of relatively small primes raised to relatively large powers. Thus the case $m = 2^s$ is a natural focuse point, being the easiest for program implementations, since the reduction of a positive integer rational modulo 2^s is merely a truncation of all its 2-base expansion senior bits, starting with the sth one (our numbering of digits starts with 0). It is this case where one may use natural (judging by program implementation) operations other than the above mentioned arithmetical ones, namely, bitwise logical operations like XOR, OR, AND and other bitwise operations with nonnegative rational integer operands represented as base-2 expansions. And, luckely, there is a complete description of measure-preserving (or ergodic) functions in this case; see section 2 of the paper.

The obtained results make it possible to construct pseudorandom number generators that meet some requirements for performance, statistical quality or cryptographical security. This theme will be thoroughly studied in forthcoming papers. Here we briefly note only that the application of equiprobable functions, which are also studied in the paper, as output functions of congruential generators with ergodic state update functions, allows us to preserve uniformity of distribution and simultaneously eliminate one more well known disadvantage of congruential generators, the so-called 'less significant bit effect'. The latter demonstrates each sequence $\{x_n\}$ that satisfy recurrence relation $x_{n+1} \equiv f(x_n) \pmod{2^k}$ with a compatible $f\colon \mathbb{Z}_2 \to \mathbb{Z}_2$: The smallest period of a sequence composed of all jth digits of each x_n is of length 2^{j+1}. Methods of remedy will also be studied in one of future papers.

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